On the theoretical foundations of thin solid and liquid shells

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Published[2] in *Mathematics and Mechanics of Solids*, DOI: 10.1177/1081286515594656
Submitted on 30. May 2015, Accepted on 13. June 2015

Abstract This paper gives a concise summary of the general theoretical framework suitable to describe shells with solid-like and liquid-like behavior. Thin shell kinematics are considered and used to derive the equilibrium equations from linear and angular momentum balance. Based on the mechanical power balance and the mechanical dissipation inequality, the constitutive equations for the hyperelastic material behavior of constrained shells are derived, and their material stability is examined. Various constitutive examples are considered and assessed for their stability. The governing weak form of the formulation is derived and decomposed into in-plane and out-of-plane components. The presented work provides a very general framework for a unified description of solid and liquid shells and illustrates what leads to their loss of material stability. This framework serves as a basis for developing computational shell formulations based on rotation-free shell discretizations. Therefore the full linearization of the formulation is also presented here.

Keywords: area incompressibility, linearization, lipid bilayers, material stability, rotation-free shell formulations, thin shell theory

1 Introduction

The aim of this work is to provide a concise yet general and unified theoretical framework for both solid and liquid shells that is suitable for their numerical description using intrinsic, rotation-free surface discretizations.

A large literature body exists on theoretical shell formulations, even for the general cases of arbitrary surface geometries, large deformation and nonlinear material behavior. The scope of these models can for example be found in the texts of Naghdi (1982), Pietraszkiewicz (1989) and Libai and Simmonds (1998), and the references cited therein. In those works the focus lies on solid shells that exhibit classical solid-like material behavior such as elasticity. In contrast to those, some shells exhibit liquid-like behavior. An important example are lipid bilayers forming cell membranes. Those membranes provide elastic resistance to bending, while the in-plane membrane behavior is that of a fluid – there is flow without static resistance. Such behavior cannot be described adequately by solid shell models. Instead, a generalization of the formulation is required that accounts for the constitutive behavior of liquid shells, such as cell membranes. In general this behavior is characterized by solid-like deformation and fluid-like flow. As a first step, we will restrict ourselves here to the quasi-static setting, and present a unified shell formulation for that. A particular focus in our presentation is to account for

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[2]This pdf is the personal version of an article whose final publication is available at http://mms.sagepub.com

the area-incompressibility of the shell, as this is a common assumption used for liquid shells. We will further restrict ourselves to thin shell theories of the Kirchhoff-Love type that neglect shear-deformations over the shell thickness. Those theories can be obtained by constraining the general shell kinematics (Steigmann, 1999b). They are interesting from the computational point of view, as they allow for straightforward, yet efficient and accurate discretization methods based on isogeometric analysis (Kiendl et al., 2009). The development of such computational methods, eventually in the context of dynamic flow, is the motivation behind this paper.
The description of liquid shells goes back to the bending models of Canham (1970) and Helfrich (1973), which have then been used in many subsequent studies. The models can be embedded into the framework of general shell theories (Steigmann, 1999a), which is the description we follow here.

Compared to earlier approaches our work contains the following novelties and merits:

- a concise theory, applied to a wide range of different constitutive models considering constrained solids and liquids,
- the investigation of the stability of those models,
- the stiffness tensors of those models, which are required for linearization,
- the weak form considering a split into in-plane and out-of-plane contributions,
- the analytical solution for a sheet under pure bending and homogeneous stretching.

The remainder of this paper is structured as follows: Sec. 2 summaries the major kinematical measures required to describe thin shells. Those are then used in Secs. 3 and 4 to derive the equilibrium and constitutive equations considering hyperelasticity. A large range of constitutive examples – considering quasi-static solid and liquid material behavior – is examined in Sec. 5. Sec. 6 then presents the weak form governing those shell models. The models are applied to a simple analytical example in Sec. 7. The paper concludes with Sec. 8.

2 Thin shell kinematics

This section discusses the kinematics of deforming surfaces and examines the influence of variations of the deformation. The general framework provided by curvilinear coordinates is considered.

2.1 Curvilinear surface description

Consider a general surface $\mathcal{S}$ embedded in 3D space. Points on the surface can be described by the mapping

$$\boldsymbol{x} = \boldsymbol{x}(\xi^\alpha) \;, \tag{1}$$

where ξ^α, $\alpha = 1, 2$ are curvilinear coordinates. Given this surface parameterization, we can obtain the co-variant tangent vectors $\boldsymbol{a}_\alpha = \partial\boldsymbol{x}/\partial\xi^\alpha$, the metric tensor with co-variant components $a_{\alpha\beta} = \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta$ and contra-variant components $[a^{\alpha\beta}] = [a_{\alpha\beta}]^{-1}$, the contra-variant tangent vectors $\boldsymbol{a}^\alpha = a^{\alpha\beta}\boldsymbol{a}_\beta$, the area change $\mathrm{d}a = J_a\,\mathrm{d}\xi^1\,\mathrm{d}\xi^2$ with $J_a = \sqrt{\det a_{\alpha\beta}}$, the surface normal $\boldsymbol{n} = (\boldsymbol{a}_1 \times \boldsymbol{a}_2)/J_a$, the parametric derivative of $\boldsymbol{a}_\alpha$, as $\boldsymbol{a}_{\alpha,\beta} = \partial\boldsymbol{a}_\alpha/\partial\xi^\beta$, and the co-variant derivative of $\boldsymbol{a}_\alpha$, as $\boldsymbol{a}_{\alpha;\beta} = (\boldsymbol{n}\otimes\boldsymbol{n})\,\boldsymbol{a}_{\alpha,\beta}$. The latter can also be defined through the Christoffel symbols $\Gamma^\gamma_{\alpha\beta} = \boldsymbol{a}^\gamma \cdot \boldsymbol{a}_{\alpha,\beta}$. The triads $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{n}\}$ and $\{\boldsymbol{a}^1, \boldsymbol{a}^2, \boldsymbol{n}\}$ form bases to decompose vectors $\boldsymbol{v} \in \mathbb{R}^3$ into their in-plane and out-of-plane components, written as

$$\boldsymbol{v} = v^\alpha\, \boldsymbol{a}_\alpha + v\, \boldsymbol{n} = v_\alpha\, \boldsymbol{a}^\alpha + v\, \boldsymbol{n} \ , \tag{2}$$

where $v = \boldsymbol{v} \cdot \boldsymbol{n}$ is the vector component along $\boldsymbol{n}$, and $v_\alpha = \boldsymbol{v} \cdot \boldsymbol{a}_\alpha$ and $v^\alpha = \boldsymbol{v} \cdot \boldsymbol{a}^\alpha$ are the co-variant and contra-variant vector components. The co-variant derivative of $\boldsymbol{v}$ is equal to the parametric one, implying the four identities $\boldsymbol{n}_{;\alpha} = \boldsymbol{n}_{,\alpha}$, $v_{;\alpha} = v_{,\alpha}$, $(v^\alpha \boldsymbol{a}_\alpha)_{;\beta} = (v^\alpha \boldsymbol{a}_\alpha)_{,\beta}$ and $(v_\alpha \boldsymbol{a}^\alpha)_{;\beta} = (v_\alpha \boldsymbol{a}^\alpha)_{,\beta}$. Similar decompositions to (2) follow for higher order tensors. Two important second order tensors are the surface identity tensor,

$$\boldsymbol{1} = \boldsymbol{a}_\alpha \otimes \boldsymbol{a}^\alpha = \boldsymbol{a}^\alpha \otimes \boldsymbol{a}_\alpha \ , \tag{3}$$

and the full identity in $\mathbb{R}^3$,

$$\boldsymbol{I} = \boldsymbol{1} + \boldsymbol{n} \otimes \boldsymbol{n} \ . \tag{4}$$

Another important object is the curvature tensor $\boldsymbol{b} = b_{\alpha\beta}\, \boldsymbol{a}^\alpha \otimes \boldsymbol{a}^\beta$ with the co-variant components

$$b_{\alpha\beta} = \boldsymbol{n} \cdot \boldsymbol{a}_{\alpha,\beta} = -\boldsymbol{n}_{,\beta} \cdot \boldsymbol{a}_\alpha \ , \tag{5}$$

the mixed components $b^\alpha_\beta = a^{\alpha\gamma}\, b_{\gamma\beta}$ and the contra-variant components $b^{\alpha\beta} = b^\alpha_\gamma\, a^{\gamma\beta}$. It appears in the formulas of Gauss,

$$\boldsymbol{a}_{\alpha;\beta} = b_{\alpha\beta}\, \boldsymbol{n} \ , \tag{6}$$

and Weingarten,

$$\boldsymbol{n}_{,\alpha} = -b^\beta_\alpha\, \boldsymbol{a}_\beta \ , \tag{7}$$

and defines the mean curvature of $\mathcal{S}$,

$$H := \frac{1}{2}\, b^\alpha_\alpha = \frac{1}{2}\, a^{\alpha\beta}\, b_{\alpha\beta} \ , \tag{8}$$

and the Gaussian curvature of $\mathcal{S}$,

$$\kappa := \frac{1}{2}\, e^{\alpha\beta} e^{\lambda\mu} b_{\alpha\lambda} b_{\beta\mu}\, \frac{1}{a} = \frac{b}{a} \ , \tag{9}$$

where

$$a = \det[a_{\alpha\beta}] \ , \tag{10}$$

$$b = \det[b_{\alpha\beta}] \ , \tag{11}$$

and

$$[e^{\alpha\beta}] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{12}$$

is the so-called unit alternator. From H and κ we can find the principal surface curvatures

$$\kappa_{1/2} = H \pm \sqrt{H^2 - \kappa} \ , \tag{13}$$

such that $2H = \kappa_1 + \kappa_2$ and $\kappa = \kappa_1\, \kappa_2$. The unit alternator is useful for inversion. If $c_{\alpha\beta}$ are the co-variant components of a tensor $\boldsymbol{c}$ that is invertible in the tangent plane, then the contra-variant components of its inverse are given by

$$c^{\alpha\beta}_{\text{inv}} = \frac{1}{c}\, e^{\alpha\gamma}\, c_{\delta\gamma}\, e^{\beta\delta} \ , \quad c = \det[c_{\alpha\beta}] \ . \tag{14}$$

In particular we thus have

$$a^{\alpha\beta} = \frac{1}{a}\, e^{\alpha\gamma}\, a_{\gamma\delta}\, e^{\beta\delta} \ . \tag{15}$$

According to the Cayley-Hamilton-theorem, the tensor $\boldsymbol{c}$ satisfies the identity

$$c^{\gamma}_{\gamma}\, a^{\alpha\beta} - c^{\alpha\beta} = \tilde{c}^{\alpha\beta} \ , \tag{16}$$

where $\tilde{c}^{\alpha\beta} := \dfrac{c}{a} c^{\alpha\beta}_{\mathrm{inv}}$ are the contra-variant components of the adjugate tensor. For the curvature tensor in particular we find

$$2H\, a^{\alpha\beta} - b^{\alpha\beta} = \kappa\, b^{\alpha\beta}_{\mathrm{inv}} \ , \tag{17}$$

$$b^{\alpha\gamma}\, b^{\beta}_{\gamma} = 2H\, b^{\alpha\beta} - \kappa\, a^{\alpha\beta} \ , \tag{18}$$

and

$$b^{\gamma}_{\alpha}\, b_{\gamma\beta} = 2H\, b_{\alpha\beta} - \kappa\, a_{\alpha\beta} \ . \tag{19}$$

From (15) we can further find

$$a = \frac{1}{2}\, e^{\alpha\gamma}\, e^{\beta\delta}\, a_{\alpha\beta}\, a_{\gamma\delta} \ . \tag{20}$$

2.2 Surface deformation

In order to describe the deformation of surface $\mathcal{S}$, we introduce a reference configuration, denoted $\mathcal{S}_0$, that will typically agree with the initial, undeformed configuration of $\mathcal{S}$. The reference configuration is described by the mapping $\boldsymbol{X} = \boldsymbol{X}(\xi^{\alpha})$. The same kinematical surface quantities introduced above can be obtained in analogy for $\boldsymbol{X} \in \mathcal{S}_0$. We will distinguish those using either capital letters or subscript '0'. The deformation map between $\mathcal{S}_0$ and $\mathcal{S}$, denoted $\boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{X})$, is characterized by the surface deformation gradient

$$\boldsymbol{F} := \boldsymbol{a}_{\alpha} \otimes \boldsymbol{A}^{\alpha} \ , \tag{21}$$

obtained from $\mathrm{d}\boldsymbol{x} = \boldsymbol{F}\,\mathrm{d}\boldsymbol{X}$, and the area change $\mathrm{d}a = J\,\mathrm{d}A$, where J denotes the surface stretch defined by $J = J_a/J_A$ and $J_A = \sqrt{\det A_{\alpha\beta}}$. If the number of surface particles is conserved during deformation, as we will consider here, we have

$$\rho\,\mathrm{d}a = \rho_0\,\mathrm{d}A \ , \tag{22}$$

such that

$$J = \frac{\rho_0}{\rho} \ , \tag{23}$$

where ρ and ρ_0 are the surface densities at $\boldsymbol{x}$ and $\boldsymbol{X}$. The left and right Cauchy-Green surface tensors follow as $\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F} = a_{\alpha\beta}\,\boldsymbol{A}^{\alpha} \otimes \boldsymbol{A}^{\beta}$ and $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T = A^{\alpha\beta}\,\boldsymbol{a}_{\alpha} \otimes \boldsymbol{a}_{\beta}$. Evaluating their trace $I_1 := \boldsymbol{C} : \boldsymbol{1}_0 = \boldsymbol{B} : \boldsymbol{1}$, where $\boldsymbol{1}_0 = \boldsymbol{A}_{\alpha} \otimes \boldsymbol{A}^{\alpha}$ analogous to Eq. (3), gives

$$I_1 = A^{\alpha\beta} a_{\alpha\beta} \ . \tag{24}$$

2.3 Variation of kinematical quantities

For the theoretical developments in the subsequent sections, the variation of several kinematical quantities is required. We therefore consider a variation of position $\boldsymbol{x}$ on surface $\mathcal{S}$ by the amount $\delta\boldsymbol{x}$ and examine how it affects various kinematical quantities. Partly similar examinations can be found in Steigmann et al. (2003) or recently in Sauer (2016). The variation of the tangent vectors and its parametric derivative become $\delta\boldsymbol{a}_{\alpha} = \delta\boldsymbol{x}_{,\alpha}$ and $\delta\boldsymbol{a}_{\alpha,\beta} = \delta\boldsymbol{x}_{,\alpha\beta}$, so that

$$\delta a_{\alpha\beta} = \boldsymbol{a}_{\alpha} \cdot \delta\boldsymbol{a}_{\beta} + \delta\boldsymbol{a}_{\alpha} \cdot \boldsymbol{a}_{\beta} \tag{25}$$

and

$$\delta b_{\alpha\beta} = \boldsymbol{a}_{\alpha,\beta} \cdot \delta\boldsymbol{n} + \boldsymbol{n} \cdot \delta\boldsymbol{a}_{\alpha,\beta} \tag{26}$$

or

$$\delta b_{\alpha\beta} = -\delta\boldsymbol{n}_{,\alpha} \cdot \boldsymbol{a}_{\beta} - \boldsymbol{n}_{,\alpha} \cdot \delta\boldsymbol{a}_{\beta} \ . \tag{27}$$

The variation of the normal is

$$\delta\boldsymbol{n} = -\boldsymbol{a}^{\alpha}(\boldsymbol{n} \cdot \delta\boldsymbol{a}_{\alpha}) = -(\boldsymbol{a}^{\alpha} \otimes \boldsymbol{n})\,\delta\boldsymbol{a}_{\alpha} \ , \tag{28}$$

e.g. see Wriggers (2006), such that

$$\delta b_{\alpha\beta} = \big(\delta\boldsymbol{a}_{\alpha,\beta} - \Gamma^{\gamma}_{\alpha\beta}\,\delta\boldsymbol{a}_{\gamma}\big) \cdot \boldsymbol{n} \ . \tag{29}$$

The variation of $\boldsymbol{a}^{\alpha}$ is

$$\delta\boldsymbol{a}^{\alpha} = \big(a^{\alpha\beta}\,\boldsymbol{n} \otimes \boldsymbol{n} - \boldsymbol{a}^{\beta} \otimes \boldsymbol{a}^{\alpha}\big)\,\delta\boldsymbol{a}_{\beta} \ , \tag{30}$$

e.g. see Sauer (2016). From Eq. (20) follows

$$\delta a = a\,a^{\alpha\beta}\,\delta a_{\alpha\beta} \ , \tag{31}$$

and therefore

$$\delta J = \frac{\partial J}{\partial a_{\alpha\beta}}\,\delta a_{\alpha\beta} = \frac{J}{2}\,a^{\alpha\beta}\,\delta a_{\alpha\beta} \ . \tag{32}$$

From Eq. (15) we get

$$\delta a^{\alpha\beta} = a^{\alpha\beta\gamma\delta}\,\delta a_{\gamma\delta} \tag{33}$$

with[3]

$$a^{\alpha\beta\gamma\delta} := \frac{\partial a^{\alpha\beta}}{\partial a_{\gamma\delta}} = \frac{1}{2a}\big(e^{\alpha\gamma}e^{\beta\delta} + e^{\alpha\delta}e^{\beta\gamma}\big) - a^{\alpha\beta}a^{\gamma\delta} \ . \tag{34}$$

From a component-wise comparison it can be further shown that

$$a^{\alpha\beta\gamma\delta} = -\frac{1}{2}\big(a^{\alpha\gamma}a^{\beta\delta} + a^{\alpha\delta}a^{\beta\gamma}\big) \ . \tag{35}$$

It is noted that $a^{\alpha\beta\gamma\delta}$ has major and minor symmetries. Contracting $a^{\alpha\beta\gamma\delta}$ with any symmetric tensor with components $c_{\gamma\delta}$, yields

$$a^{\alpha\beta\gamma\delta}\,c_{\gamma\delta} = -c^{\alpha\beta} \ . \tag{36}$$

For this particular operation $a^{\alpha\beta\gamma\delta}$ may then be replaced by $-a^{\alpha\gamma}a^{\beta\delta}$.
The variation of the mean curvature yields

$$\delta H = \frac{1}{2}\,\delta a^{\alpha\beta}\,b_{\alpha\beta} + \frac{1}{2}\,a^{\alpha\beta}\,\delta b_{\alpha\beta} \ . \tag{37}$$

Using Eqs. (33) and (36) gives

$$\delta H = \frac{\partial H}{\partial a_{\alpha\beta}}\,\delta a_{\alpha\beta} + \frac{\partial H}{\partial b_{\alpha\beta}}\,\delta b_{\alpha\beta} \ , \tag{38}$$

with

$$\begin{aligned} \frac{\partial H}{\partial a_{\alpha\beta}} &= -\frac{1}{2}b^{\alpha\beta} \ , \\ \frac{\partial H}{\partial b_{\alpha\beta}} &= \frac{1}{2}a^{\alpha\beta} \ . \end{aligned} \tag{39}$$

[3] In previous papers (Sauer et al., 2014; Sauer, 2014, 2016) we had used the quantity $m^{\alpha\beta\gamma\delta} = 2a^{\alpha\beta\gamma\delta}$.

The variation of the Gaussian curvature gives

$$\delta\kappa = \frac{1}{2}\, e^{\alpha\beta} e^{\lambda\mu}\, \delta b_{\alpha\lambda}\, b_{\beta\mu}\, \frac{1}{a} + \frac{1}{2}\, e^{\alpha\beta} e^{\lambda\mu}\, b_{\alpha\lambda}\, \delta b_{\beta\mu}\, \frac{1}{a} - \frac{1}{2}\, e^{\alpha\beta} e^{\lambda\mu}\, b_{\alpha\lambda}\, b_{\beta\mu}\, \frac{1}{a^2}\, \delta a \;. \tag{40}$$

Using (9), (14) and (31), this can be rewritten into

$$\delta\kappa = \frac{\partial\kappa}{\partial a_{\alpha\beta}}\, \delta a_{\alpha\beta} + \frac{\partial\kappa}{\partial b_{\alpha\beta}}\, \delta b_{\alpha\beta} \;, \tag{41}$$

with

$$\begin{aligned} \frac{\partial\kappa}{\partial a_{\alpha\beta}} &= -\kappa\, a^{\alpha\beta} \;, \\ \frac{\partial\kappa}{\partial b_{\alpha\beta}} &= \kappa\, b^{\alpha\beta}_{\text{inv}} = \tilde{b}^{\alpha\beta} \;. \end{aligned} \tag{42}$$

Further, we will need $\delta b^{\alpha\beta}$. Taking the variation of $b^{\alpha\beta} = b_{\gamma\delta}\, a^{\gamma\alpha}\, a^{\delta\beta}$ yields

$$\delta b^{\alpha\beta} = b^{\alpha\beta\gamma\delta}\, \delta a_{\gamma\delta} - a^{\alpha\beta\gamma\delta}\, \delta b_{\gamma\delta} \;, \tag{43}$$

with

$$b^{\alpha\beta\gamma\delta} := -\frac{1}{2}\Big(a^{\alpha\gamma}\, b^{\beta\delta} + b^{\alpha\gamma}\, a^{\beta\delta} + a^{\alpha\delta}\, b^{\beta\gamma} + b^{\alpha\delta}\, a^{\beta\gamma}\Big) \;. \tag{44}$$

From a component-wise comparison, it can be shown that $b^{\alpha\beta\gamma\delta}$ is also equal to

$$b^{\alpha\beta\gamma\delta} = 2H\left(a^{\alpha\beta}\, a^{\gamma\delta} + a^{\alpha\beta\gamma\delta}\right) - \left(a^{\alpha\beta}\, b^{\gamma\delta} + b^{\alpha\beta}\, a^{\gamma\delta}\right) \;. \tag{45}$$

From Eq. (43) we can then identify the derivatives

$$\begin{aligned} \frac{\partial b^{\alpha\beta}}{\partial a_{\gamma\delta}} &= b^{\alpha\beta\gamma\delta} \;, \\ \frac{\partial b^{\alpha\beta}}{\partial b_{\gamma\delta}} &= -a^{\alpha\beta\gamma\delta} \;. \end{aligned} \tag{46}$$

3 Equilibrium

This section discusses the equilibrium and balance conditions for shells. As an intial step we need to introduce sectional forces and sectional moments.

3.1 Sectional forces and moments

Consider an infinitesimal surface element $\mathrm{d}a$, located at $\boldsymbol{x}$ and aligned along $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ as is shown in Fig. 1. On the cut surfaces the distributed[4] sectional force and moment components $N^{\alpha\beta}$, S^{α} and $M^{\alpha\beta}$ are defined as shown. The sectional forces are collected in the stress tensor

$$\boldsymbol{\sigma} := N^{\alpha\beta}\, \boldsymbol{a}_\alpha \otimes \boldsymbol{a}_\beta + S^{\alpha}\, \boldsymbol{a}_\alpha \otimes \boldsymbol{n} \;, \tag{47}$$

such that the traction vector on the cut normal to $\boldsymbol{\nu}$ is given through Cauchy's formula

$$\boldsymbol{T} := \boldsymbol{\sigma}^{\mathrm{T}} \boldsymbol{\nu} \;. \tag{48}$$

[4]per current length of the cut face

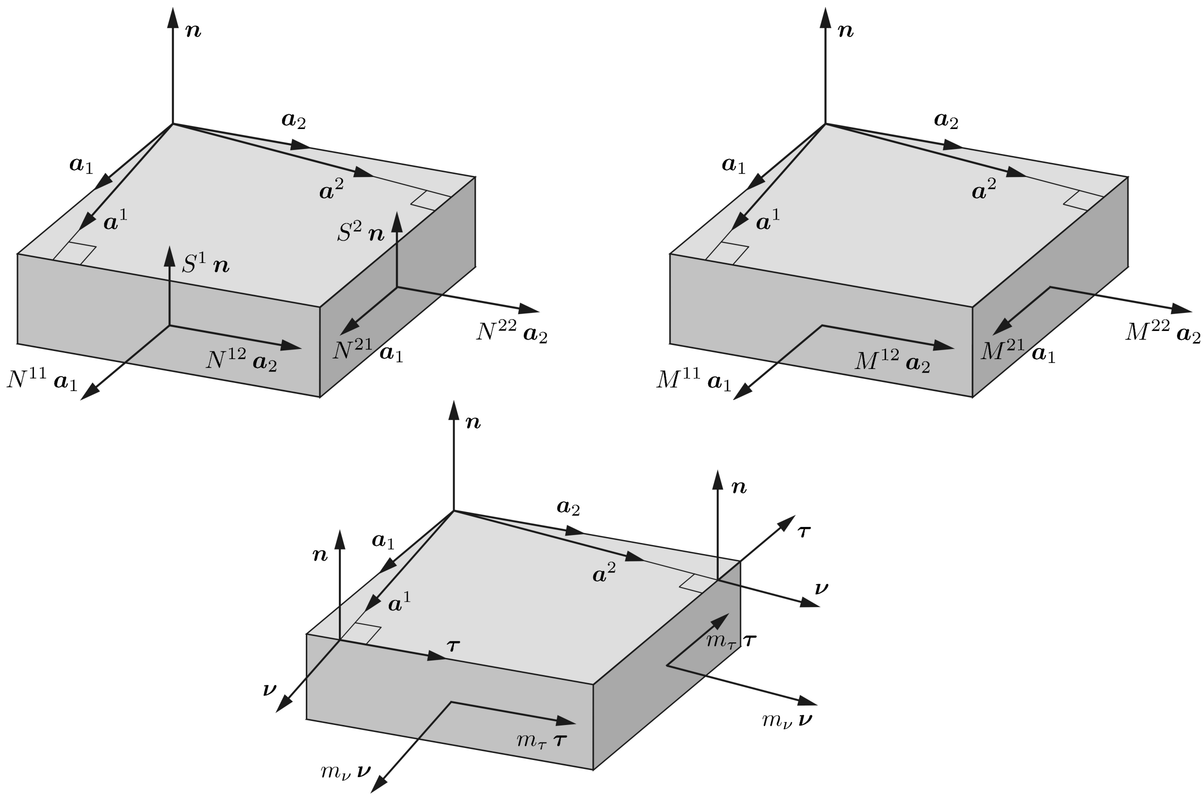

Figure 1: Components of the traction and moment vectors $\boldsymbol{T}^1$, $\boldsymbol{T}^2$, $\boldsymbol{M}^1$ and $\boldsymbol{M}^2$ defined on the faces normal to $\boldsymbol{a}^1$ and $\boldsymbol{a}^2$ (top). Components of the physical moment vector $\boldsymbol{m}$ acting on the same faces (bottom).

With $\boldsymbol{\nu} = \nu_\alpha\, \boldsymbol{a}^\alpha$ we can write $\boldsymbol{T} = \boldsymbol{T}^\alpha\, \nu_\alpha$, where

$$\boldsymbol{T}^\alpha := \boldsymbol{\sigma}^{\mathrm{T}}\, \boldsymbol{a}^\alpha = N^{\alpha\beta}\, \boldsymbol{a}_\beta + S^\alpha\, \boldsymbol{n} \ , \tag{49}$$

are then the tractions defined on the face normal to $\boldsymbol{a}^\alpha$, see Fig. 1.
The distributed section moments are collected in the moment tensor

$$\boldsymbol{\mu} := -M^{\alpha\beta}\, \boldsymbol{a}_\alpha \otimes \boldsymbol{a}_\beta \ , \tag{50}$$

such that we can define the distributed moment vector

$$\boldsymbol{M} := \boldsymbol{\mu}^{\mathrm{T}}\, \boldsymbol{\nu} \tag{51}$$

on the cut normal to $\boldsymbol{\nu}$. Similar to before we can write

$$\boldsymbol{M} = \boldsymbol{M}^\alpha\, \nu_\alpha \ , \tag{52}$$

with

$$\boldsymbol{M}^\alpha := \boldsymbol{\mu}^{\mathrm{T}}\, \boldsymbol{a}^\alpha = -M^{\alpha\beta}\, \boldsymbol{a}_\beta \ . \tag{53}$$

The components of $-\boldsymbol{M}^\alpha$ are shown in the top right inset of Fig. 1. Vector $\boldsymbol{M}$ is introduced for convenience. The moment vector physically acting on the element is given be the rotated quantity

$$\boldsymbol{m} := \boldsymbol{n} \times \boldsymbol{M} \ . \tag{54}$$

Inserting (52) and (53), and using the identity

$$\boldsymbol{a}_\beta \times \boldsymbol{n} = \tau_\beta\, \boldsymbol{\nu} - \nu_\beta\, \boldsymbol{\tau} \ , \tag{55}$$

we find

$$\boldsymbol{m} = m_\nu\,\boldsymbol{\nu} + m_\tau\,\boldsymbol{\tau} \tag{56}$$

with the local Cartesian components

$$\begin{aligned} m_\nu &:= M^{\alpha\beta}\,\nu_\alpha\,\tau_\beta\ , \\ m_\tau &:= -M^{\alpha\beta}\,\nu_\alpha\,\nu_\beta\ . \end{aligned} \tag{57}$$

The vector $\boldsymbol{M}$ can then also be written as

$$\boldsymbol{M} = m_\tau\,\boldsymbol{\nu} - m_\nu\,\boldsymbol{\tau}\ . \tag{58}$$

The bottom inset of Fig. 1 shows the vector $\boldsymbol{m}$ acting on faces $\boldsymbol{a}^\alpha$.

3.2 Balance of linear momentum

Let us now consider a part of the surface $\mathcal{S}$ denoted $\mathcal{P}$. We denote the body forces (per current surface area) acting on $\mathcal{P}$ by $\boldsymbol{f}$, and we assume that the boundary of $\mathcal{P}$ is smooth. For every such surface part, the change of its linear momentum is equal to the external forces acting on it, i.e.

$$\frac{D}{Dt}\int_{\mathcal{P}} \rho\,\boldsymbol{v}\,\mathrm{d}a = \int_{\mathcal{P}} \boldsymbol{f}\,\mathrm{d}a + \int_{\partial\mathcal{P}} \boldsymbol{T}\,\mathrm{d}s \quad \forall\,\mathcal{P}\subset\mathcal{S}\ . \tag{59}$$

Here D/Dt denotes the material time derivative, and $\boldsymbol{v}$ is the material velocity at $\boldsymbol{x}$. With the local conservation of mass (22) and Stokes theorem

$$\int_{\partial\mathcal{P}} \boldsymbol{T}^\alpha\,\nu_\alpha\,\mathrm{d}s = \int_{\mathcal{P}} \boldsymbol{T}^\alpha_{;\alpha}\,\mathrm{d}a\ , \tag{60}$$

we immediately arrive at the local form of (59),

$$\boldsymbol{T}^\alpha_{;\alpha} + \boldsymbol{f} = \rho\,\dot{\boldsymbol{v}} \quad \forall\,\boldsymbol{x}\in\mathcal{S}\ , \tag{61}$$

which is the strong form equilibrium equation at $\boldsymbol{x}\in\mathcal{S}$. It can be decomposed into in-plane and out-of-plane contributions by using (49) to find

$$\boldsymbol{T}^\gamma_{;\gamma} = (N^{\gamma\alpha}_{\ \ ;\gamma} - b^\alpha_\gamma\,S^\gamma)\,\boldsymbol{a}_\alpha + (S^\gamma_{;\gamma} + b_{\alpha\gamma}N^{\alpha\gamma})\,\boldsymbol{n}\ . \tag{62}$$

Decomposing the external force $\boldsymbol{f} = f^\alpha\,\boldsymbol{a}_\alpha + p\,\boldsymbol{n}$ and material acceleration $\dot{\boldsymbol{v}} := \boldsymbol{a} = a^\alpha\,\boldsymbol{a}_\alpha + a_\mathrm{n}\,\boldsymbol{n}$ we then find the in-plane equilibrium equation

$$N^{\gamma\alpha}_{\ \ ;\gamma} - b^\alpha_\gamma\,S^\gamma + f^\alpha = \rho\,a^\alpha\ , \tag{63}$$

and the out-of-plane equilibrium equation

$$S^\gamma_{;\gamma} + N^{\alpha\gamma}\,b_{\alpha\gamma} + p = \rho\,a_\mathrm{n}\ . \tag{64}$$

For an alternative derivation of these equations, see for example Jenkins (1977).

3.3 Balance of angular momentum

For every surface part $\mathcal{P} \subset \mathcal{S}$, the change of angular momentum is equal to the moment of the external forces, i.e.

$$\frac{D}{Dt}\int_{\mathcal{P}} \rho\,\boldsymbol{x}\times\boldsymbol{v}\,\mathrm{d}a = \int_{\mathcal{P}} \boldsymbol{x}\times\boldsymbol{f}\,\mathrm{d}a + \int_{\partial\mathcal{P}} \boldsymbol{x}\times\boldsymbol{T}\,\mathrm{d}s + \int_{\partial\mathcal{P}} \boldsymbol{m}\,\mathrm{d}s \quad \forall\mathcal{P}\subset\mathcal{S}\,. \tag{65}$$

As usual for mass conservation

$$\frac{D}{Dt}\int_{\mathcal{P}} \rho\,\boldsymbol{x}\times\boldsymbol{v}\,\mathrm{d}a = \int_{\mathcal{P}} \rho\,\boldsymbol{x}\times\dot{\boldsymbol{v}}\,\mathrm{d}a\,. \tag{66}$$

Stokes theorem now gives

$$\int_{\partial\mathcal{P}} \boldsymbol{x}\times\boldsymbol{T}\,\mathrm{d}s = \int_{\mathcal{P}} \Big(\boldsymbol{a}_\alpha\times\boldsymbol{T}^\alpha + \boldsymbol{x}\times\boldsymbol{T}^\alpha_{;\alpha}\Big)\,\mathrm{d}a\,, \tag{67}$$

and, due to (54), (52) and (53),

$$\int_{\partial\mathcal{P}} \boldsymbol{m}\,\mathrm{d}s = \int_{\mathcal{P}} (M^{\beta\alpha}\,\boldsymbol{a}_\alpha\times\boldsymbol{n})_{;\beta}\,\mathrm{d}a\,. \tag{68}$$

Using (6) and (7), the last equation expands into

$$\int_{\partial\mathcal{P}} \boldsymbol{m}\,\mathrm{d}s = \int_{\mathcal{P}} \Big(M^{\beta\alpha}_{\ \ ;\beta}\,\boldsymbol{a}_\alpha\times\boldsymbol{n} - b^\beta_\gamma\,M^{\gamma\alpha}\,\boldsymbol{a}_\alpha\times\boldsymbol{a}_\beta\Big)\,\mathrm{d}a\,. \tag{69}$$

Employing (61) and (49) and using the equations (66), (67) and (69) from above, the balance of angular momentum (65) thus gives

$$\int_{\mathcal{P}} \boldsymbol{a}_\alpha\times\Big[\big(N^{\alpha\beta} - b^\beta_\gamma\,M^{\gamma\alpha}\big)\,\boldsymbol{a}_\beta + \big(S^\alpha + M^{\beta\alpha}_{\ \ ;\beta}\big)\,\boldsymbol{n}\Big]\,\mathrm{d}a = \boldsymbol{0} \quad \forall\mathcal{P}\subset\mathcal{S}\,. \tag{70}$$

This is satisfied if and only if

$$\sigma^{\alpha\beta} := N^{\alpha\beta} - b^\beta_\gamma\,M^{\gamma\alpha} \tag{71}$$

is symmetric and

$$S^\alpha = -M^{\beta\alpha}_{\ \ ;\beta}\,. \tag{72}$$

The last equation expresses the well known Kirchhoff-Love result that the out-of-plane shear component follows as the derivative of the bending moments. It turns out that apart from $\sigma^{\alpha\beta}$ also $M^{\alpha\beta}$ is symmetric, see Sec. 4.3. According to relation (71), the in-plane stress component

$$N^{\alpha\beta} = \sigma^{\alpha\beta} + b^\beta_\gamma\,M^{\gamma\alpha} \tag{73}$$

is influenced by bending. Due to this, $N^{\alpha\beta}$ is generally not symmetric. This influence is a high order effect, that vanishes for flat plates when $b^\beta_\gamma \to 0$. It also vanishes for very thin shells, since, even for large deformations, typically $\sigma^\alpha_\beta \sim ET$ while $M^\alpha_\beta \sim ET^3$, where E is Young's modulus and T is the shell thickness. This is illustrated further in the example of Sec. 7.

3.4 On the application of boundary moments and tractions

At the boundary of the shell surface, essential and natural boundary conditions – the latter for tractions and bending moments – can be applied. It is well known that Kirchhoff-Love shells cannot support independent boundary moments and tractions. The moment $\boldsymbol{m}_\nu$, introduced in

(56), is perceived as an effective traction acting together with $\boldsymbol{T}$ at the boundary. This can be seen by looking at the integral of $\boldsymbol{m}$ appearing in (65). We therefore note that since $\boldsymbol{\nu} = \boldsymbol{\tau} \times \boldsymbol{n}$ and $\boldsymbol{\tau} = \partial \boldsymbol{x}/\partial s$, we have

$$m_\nu \, \boldsymbol{\nu} = (m_\nu \, \boldsymbol{x} \times \boldsymbol{n})' - \boldsymbol{x} \times (m_\nu \, \boldsymbol{n})' \, , \tag{74}$$

where $(...)' := \partial .../\partial s$. The first part integrates to zero over a smooth and closed boundary, such that we obtain

$$\int_{\partial\mathcal{P}} \big(\boldsymbol{x} \times \boldsymbol{T} + \boldsymbol{m} \big) \, \mathrm{d}s = \int_{\partial\mathcal{P}} \big(\boldsymbol{x} \times \boldsymbol{t} + m_\tau \, \boldsymbol{\tau} \big) \, \mathrm{d}s \, , \tag{75}$$

with the effective traction

$$\boldsymbol{t} := \boldsymbol{T} - (m_\nu \, \boldsymbol{n})' \, . \tag{76}$$

Over a non-smooth boundary containing corners, the second part in $\boldsymbol{t}$ will integrate to point loads acting at the corners, e.g. see Steigmann (1999b).
Altogether, the following boundary conditions can thus be prescribed on the shell boundary $\partial\mathcal{S} = \partial_x\mathcal{S} \cup \partial_t\mathcal{S} \cup \partial_m\mathcal{S}$

$$\begin{aligned} \boldsymbol{x} &= \bar{\boldsymbol{\varphi}} && \text{on } \partial_x\mathcal{S} \, , \\ \boldsymbol{t} &= \bar{\boldsymbol{t}} && \text{on } \partial_t\mathcal{S} \, , \\ m_\tau &= \bar{m}_\tau && \text{on } \partial_m\mathcal{S} \, , \end{aligned} \tag{77}$$

where $\bar{\boldsymbol{\varphi}}(\boldsymbol{X})$, $\bar{\boldsymbol{t}}(\boldsymbol{X})$ and $\bar{m}_\tau(\boldsymbol{X})$ denote the prescribed boundary fields.

3.5 Membranes

For pure membranes, $M^{\alpha\beta} = 0$, such that $N^{\alpha\beta} = \sigma^{\alpha\beta}$, $S^\alpha = 0$ and $\boldsymbol{t} = \boldsymbol{T}$. The stress and traction state is then characterized by

$$\boldsymbol{\sigma} = \sigma^{\alpha\beta} \, \boldsymbol{a}_\alpha \otimes \boldsymbol{a}_\beta \tag{78}$$

and

$$\boldsymbol{T}^\alpha = \sigma^{\alpha\beta} \, \boldsymbol{a}_\beta \, . \tag{79}$$

Equilibrium is still given by (61). Also (63) and (64) still apply but can be simplified (Sauer et al., 2014).

4 Shell constitution

This section discusses the constitutive framework of hyperelastic shells. Based on the mechanical power balance and the dissipation inequality, we can derive the stored energy function governing shells. Its linearization and stability are then discussed.

4.1 Mechanical power balance

The mechanical power balance follows from equilibrium. Contracting with the velocity $\boldsymbol{v}$ and integrating over $\mathcal{P} \subset \mathcal{S}$, we find

$$\int_{\mathcal{P}} \boldsymbol{v} \cdot \big(\boldsymbol{T}^\alpha_{;\alpha} + \boldsymbol{f} - \rho \, \dot{\boldsymbol{v}} \big) \, \mathrm{d}a = 0 \quad \forall \mathcal{P} \subset \mathcal{S} \, . \tag{80}$$

In here, the last term corresponds to the change of the kinetic energy

$$K := \frac{1}{2}\int_{\mathcal{P}} \rho\, \boldsymbol{v}\cdot\boldsymbol{v}\,\mathrm{d}a\ , \tag{81}$$

which, due to mass balance, is given by

$$\dot{K} := \int_{\mathcal{P}} \rho\, \boldsymbol{v}\cdot\dot{\boldsymbol{v}}\,\mathrm{d}a\ . \tag{82}$$

Using Stokes theorem, the first part can be rewritten into

$$\int_{\mathcal{P}} \boldsymbol{v}\cdot\boldsymbol{T}^{\alpha}_{;\alpha}\,\mathrm{d}a \ = \int_{\partial\mathcal{P}} \boldsymbol{v}\cdot\boldsymbol{T}^{\alpha}\,\nu_{\alpha}\,\mathrm{d}s - \int_{\mathcal{P}} \dot{\boldsymbol{a}}_{\alpha}\cdot\boldsymbol{T}^{\alpha}\,\mathrm{d}a\ , \tag{83}$$

where $\boldsymbol{T}^{\alpha}\,\nu_{\alpha} = \boldsymbol{T}$. Using Eqs. (49), (72), (7) and the property that $\sigma^{\alpha\beta}$ is symmetric, the last term becomes

$$\dot{\boldsymbol{a}}_{\alpha}\cdot\boldsymbol{T}^{\alpha} = \tfrac{1}{2}\sigma^{\alpha\beta}\,\dot{a}_{\alpha\beta} - M^{\alpha\beta}\,\boldsymbol{n}_{,\alpha}\cdot\dot{\boldsymbol{a}}_{\beta} - M^{\beta\alpha}_{\ \ ;\beta}\dot{\boldsymbol{a}}_{\alpha}\cdot\boldsymbol{n}\ . \tag{84}$$

Here, the last term now gives

$$M^{\beta\alpha}_{\ \ ;\beta}\,\dot{\boldsymbol{a}}_{\alpha}\cdot\boldsymbol{n} = \big(\dot{\boldsymbol{n}}\cdot\boldsymbol{M}^{\alpha}\big)_{;\alpha} + M^{\alpha\beta}\,\dot{\boldsymbol{n}}_{,\alpha}\cdot\boldsymbol{a}_{\beta}\ , \tag{85}$$

since $\dot{\boldsymbol{n}}\cdot\boldsymbol{a}_{\beta;\alpha} = 0$ due to (6) and (28). Inserting (85) and (84) into (83) and applying Stokes theorem and (27), we thus find

$$\int_{\mathcal{P}} \dot{\boldsymbol{a}}_{\alpha}\cdot\boldsymbol{T}^{\alpha}\,\mathrm{d}a = \frac{1}{2}\int_{\mathcal{P}} \sigma^{\alpha\beta}\,\dot{a}_{\alpha\beta}\,\mathrm{d}a + \int_{\mathcal{P}} M^{\alpha\beta}\,\dot{b}_{\alpha\beta}\,\mathrm{d}a - \int_{\partial\mathcal{P}} \dot{\boldsymbol{n}}\cdot\boldsymbol{M}\,\mathrm{d}s\ , \tag{86}$$

so that the mechanical power balance can be expressed as

$$\dot{K} + P_{\mathrm{int}} = P_{\mathrm{ext}} \quad \forall\,\mathcal{P}\subset\mathcal{S}\ , \tag{87}$$

where

$$P_{\mathrm{int}} = \frac{1}{2}\int_{\mathcal{P}} \sigma^{\alpha\beta}\,\dot{a}_{\alpha\beta}\,\mathrm{d}a + \int_{\mathcal{P}} M^{\alpha\beta}\,\dot{b}_{\alpha\beta}\,\mathrm{d}a \tag{88}$$

is the interal stress power of $\mathcal{P}$ and

$$P_{\mathrm{ext}} = \int_{\mathcal{P}} \boldsymbol{v}\cdot\boldsymbol{f}\,\mathrm{d}a + \int_{\partial\mathcal{P}} \boldsymbol{v}\cdot\boldsymbol{T}\,\mathrm{d}s + \int_{\partial\mathcal{P}} \dot{\boldsymbol{n}}\cdot\boldsymbol{M}\,\mathrm{d}s \tag{89}$$

is the power of the external forces acting on $\mathcal{P}$ and $\partial\mathcal{P}$.
The weak form of Eq. (61) can be derived analogously, see Sec. 6.

4.2 Dissipation inequality

The internal stress power (88) can be rewritten into

$$P_{\mathrm{int}} = \frac{1}{2}\int_{\mathcal{P}_0} \tau^{\alpha\beta}\,\dot{a}_{\alpha\beta}\,\mathrm{d}A + \int_{\mathcal{P}_0} M_0^{\alpha\beta}\,\dot{b}_{\alpha\beta}\,\mathrm{d}A\ , \tag{90}$$

where we have defined

$$\begin{aligned}\tau^{\alpha\beta} &:= J\sigma^{\alpha\beta}\ ,\\ M_0^{\alpha\beta} &:= JM^{\alpha\beta}\ .\end{aligned} \tag{91}$$

The local power density $\tau^{\alpha\beta}\,\dot{a}_{\alpha\beta}/2 + M_0^{\alpha\beta}\,\dot{b}_{\alpha\beta}$ appears in the mechanical dissipation inequality

$$\frac{1}{2}\tau^{\alpha\beta}\,\dot{a}_{\alpha\beta} + M_0^{\alpha\beta}\,\dot{b}_{\alpha\beta} - \dot{\Psi} \geq 0\ , \tag{92}$$

where Ψ is the Helmholtz free energy (per reference area). (92) is a consequence of the second law of thermodynamics for isothermal systems.

4.3 Constrained hyperelasticity

Let us now consider a conservative system at fixed temperature (such that $\dot{\Psi} = \dot{W}$) undergoing a cyclic process. The mechanical dissipation in every cycle must be zero, i.e.

$$\frac{1}{2}\tau^{\alpha\beta}\,\dot{a}_{\alpha\beta} + M_0^{\alpha\beta}\,\dot{b}_{\alpha\beta} - \dot{W} = 0 \tag{93}$$

for all processes. The system may be constrained, such that the stored energy density of the shell $\mathcal{S}$ can be expressed as

$$W = W_{\mathrm{x}} + W_{\mathrm{g}}\;, \tag{94}$$

where

$$W_{\mathrm{x}} = W_{\mathrm{x}}(a_{\alpha\beta}, b_{\alpha\beta}) \tag{95}$$

denotes the contribution from deformation, and

$$W_{\mathrm{g}} = q\,g(a_{\alpha\beta}, b_{\alpha\beta}) \tag{96}$$

denotes the contribution associated with a constraint $g = 0$; q is then the Lagrange multiplier associated with the constraint. Applying the chain rule then yields

$$\dot{W} = \frac{\partial W}{\partial a_{\alpha\beta}}\,\dot{a}_{\alpha\beta} + \frac{\partial W}{\partial b_{\alpha\beta}}\,\dot{b}_{\alpha\beta} + g\,\dot{q}\;, \tag{97}$$

so that (93) yields

$$\left(\frac{1}{2}\tau^{\alpha\beta} - \frac{\partial W}{\partial a^{\alpha\beta}}\right)\dot{a}_{\alpha\beta} + \left(M_0^{\alpha\beta} - \frac{\partial W}{\partial b^{\alpha\beta}}\right)\dot{b}_{\alpha\beta} - g\,\dot{q} = 0\;. \tag{98}$$

Since this applies to all processes, the usual argumentation (Coleman and Noll, 1964) leads to the constitutive equations

$$\begin{aligned} \tau^{\alpha\beta} &= 2\,\frac{\partial W}{\partial a_{\alpha\beta}} = 2\,\frac{\partial W_{\mathrm{x}}}{\partial a_{\alpha\beta}} + 2q\,\frac{\partial g}{\partial a_{\alpha\beta}}\;, \\ M_0^{\alpha\beta} &= \frac{\partial W}{\partial b_{\alpha\beta}} = \frac{\partial W_{\mathrm{x}}}{\partial b_{\alpha\beta}} + q\,\frac{\partial g}{\partial b_{\alpha\beta}}\;, \end{aligned} \tag{99}$$

and $g = 0$.

For the later developments we require the variation of W. Similar to (97), this can be written as

$$\delta W = \delta_{\mathrm{x}} W + g\,\delta q\;, \tag{100}$$

with

$$\delta_{\mathrm{x}} W := \frac{\partial W}{\partial a_{\alpha\beta}}\,\delta a_{\alpha\beta} + \frac{\partial W}{\partial b_{\alpha\beta}}\,\delta b_{\alpha\beta}\;. \tag{101}$$

Due to (99) we then have

$$\delta_{\mathrm{x}} W = \tfrac{1}{2}\,\tau^{\alpha\beta}\,\delta a_{\alpha\beta} + M_0^{\alpha\beta}\,\delta b_{\alpha\beta}\;. \tag{102}$$

If no constraint is present q and δq are considered zero.

4.4 Linearization of W

Linearizing (100), we obtain

$$\Delta\delta W = \Delta_{\mathrm{x}}\delta_{\mathrm{x}} W + \delta g\,\Delta q + \delta q\,\Delta g\;, \tag{103}$$

with

$$\delta g = \frac{\partial g}{\partial a_{\alpha\beta}}\delta a_{\alpha\beta} + \frac{\partial g}{\partial b_{\alpha\beta}}\delta b_{\alpha\beta}\;, \qquad \Delta g = \frac{\partial g}{\partial a_{\alpha\beta}}\Delta a_{\alpha\beta} + \frac{\partial g}{\partial b_{\alpha\beta}}\Delta b_{\alpha\beta}\;, \tag{104}$$

and

$$\begin{aligned}\Delta_{\mathrm{x}}\delta_{\mathrm{x}} W \;=\; & \delta a_{\alpha\beta}\frac{\partial^2 W}{\partial a_{\alpha\beta}\,\partial a_{\gamma\delta}}\Delta a_{\gamma\delta} \;+\; \delta a_{\alpha\beta}\frac{\partial^2 W}{\partial a_{\alpha\beta}\,\partial b_{\gamma\delta}}\Delta b_{\gamma\delta} \;+\; \frac{\partial W}{\partial a_{\alpha\beta}}\Delta\delta a_{\alpha\beta} \\ & +\; \delta b_{\alpha\beta}\frac{\partial^2 W}{\partial b_{\alpha\beta}\,\partial a_{\gamma\delta}}\Delta a_{\gamma\delta} \;+\; \delta b_{\alpha\beta}\frac{\partial^2 W}{\partial b_{\alpha\beta}\,\partial b_{\gamma\delta}}\Delta b_{\gamma\delta} \;+\; \frac{\partial W}{\partial b_{\alpha\beta}}\Delta\delta b_{\alpha\beta}\;.\end{aligned} \tag{105}$$

Let us introduce

$$\begin{aligned} c^{\alpha\beta\gamma\delta} &:= 4\frac{\partial^2 W}{\partial a_{\alpha\beta}\,\partial a_{\gamma\delta}} = 2\frac{\partial \tau^{\alpha\beta}}{\partial a_{\gamma\delta}}\;, \\ d^{\alpha\beta\gamma\delta} &:= 2\frac{\partial^2 W}{\partial a_{\alpha\beta}\,\partial b_{\gamma\delta}} = \frac{\partial \tau^{\alpha\beta}}{\partial b_{\gamma\delta}}\;, \\ e^{\alpha\beta\gamma\delta} &:= 2\frac{\partial^2 W}{\partial b_{\alpha\beta}\,\partial a_{\gamma\delta}} = 2\frac{\partial M_0^{\alpha\beta}}{\partial a_{\gamma\delta}}\;, \\ f^{\alpha\beta\gamma\delta} &:= \frac{\partial^2 W}{\partial b_{\alpha\beta}\,\partial b_{\gamma\delta}} = \frac{\partial M_0^{\alpha\beta}}{\partial b_{\gamma\delta}}\;, \end{aligned} \tag{106}$$

such that

$$\begin{aligned}\Delta_{\mathrm{x}}\delta_{\mathrm{x}} W \;=\; & c^{\alpha\beta\gamma\delta}\,\tfrac{1}{2}\delta a_{\alpha\beta}\,\tfrac{1}{2}\Delta a_{\gamma\delta} \;+\; d^{\alpha\beta\gamma\delta}\,\tfrac{1}{2}\delta a_{\alpha\beta}\,\Delta b_{\gamma\delta} \;+\; \tau^{\alpha\beta}\,\tfrac{1}{2}\Delta\delta a_{\alpha\beta} \\ & +\; e^{\alpha\beta\gamma\delta}\,\delta b_{\alpha\beta}\,\tfrac{1}{2}\Delta a_{\gamma\delta} \;+\; f^{\alpha\beta\gamma\delta}\,\delta b_{\alpha\beta}\,\Delta b_{\gamma\delta} \;+\; M_0^{\alpha\beta}\,\Delta\delta b_{\alpha\beta}\;.\end{aligned} \tag{107}$$

We note that $c^{\alpha\beta\gamma\delta}$ and $f^{\alpha\beta\gamma\delta}$ posses both minor and major symmetries; $d^{\alpha\beta\gamma\delta}$ and $e^{\alpha\beta\gamma\delta}$ posses only minor symmetries, but additionally we have

$$d^{\alpha\beta\gamma\delta} = e^{\gamma\delta\alpha\beta}\;. \tag{108}$$

Due to the symmetries of c, d and e, and due to Eqs. (25) and (29) we find

$$\begin{aligned} c^{\alpha\beta\gamma\delta}\,\tfrac{1}{2}\delta a_{\alpha\beta}\,\tfrac{1}{2}\Delta a_{\gamma\delta} &= \delta\boldsymbol{a}_\alpha\cdot\boldsymbol{a}_\beta\,c^{\alpha\beta\gamma\delta}\,\boldsymbol{a}_\gamma\cdot\Delta\boldsymbol{a}_\delta\;, \\ d^{\alpha\beta\gamma\delta}\,\tfrac{1}{2}\delta a_{\alpha\beta}\,\Delta b_{\gamma\delta} &= \delta\boldsymbol{a}_\alpha\cdot\boldsymbol{a}_\beta\,d^{\alpha\beta\gamma\delta}\,\boldsymbol{n}\cdot\big(\Delta\boldsymbol{a}_{\gamma,\delta} - \Gamma^\epsilon_{\gamma\delta}\,\Delta\boldsymbol{a}_\epsilon\big)\;, \\ e^{\alpha\beta\gamma\delta}\,\delta b_{\alpha\beta}\,\tfrac{1}{2}\Delta a_{\gamma\delta} &= \big(\delta\boldsymbol{a}_{\alpha,\beta} - \Gamma^\epsilon_{\alpha\beta}\,\delta\boldsymbol{a}_\epsilon\big)\cdot\boldsymbol{n}\;e^{\alpha\beta\gamma\delta}\,\boldsymbol{a}_\gamma\cdot\Delta\boldsymbol{a}_\delta\;, \\ f^{\alpha\beta\gamma\delta}\,\delta b_{\alpha\beta}\,\Delta b_{\gamma\delta} &= \big(\delta\boldsymbol{a}_{\alpha,\beta} - \Gamma^\epsilon_{\alpha\beta}\,\delta\boldsymbol{a}_\epsilon\big)\cdot\boldsymbol{n}\,f^{\alpha\beta\gamma\delta}\,\boldsymbol{n}\cdot\big(\Delta\boldsymbol{a}_{\gamma,\delta} - \Gamma^\zeta_{\gamma\delta}\,\Delta\boldsymbol{a}_\zeta\big)\;. \end{aligned} \tag{109}$$

We further have

$$\begin{aligned}\Delta\delta a_{\alpha\beta} &= \delta\boldsymbol{a}_\alpha\cdot\Delta\boldsymbol{a}_\beta + \delta\boldsymbol{a}_\beta\cdot\Delta\boldsymbol{a}_\alpha\;, \\ \Delta\delta b_{\alpha\beta} &= \delta\boldsymbol{a}_{\alpha,\beta}\cdot\Delta\boldsymbol{n} + \delta\boldsymbol{n}\cdot\Delta\boldsymbol{a}_{\alpha,\beta} + \boldsymbol{a}_{\alpha,\beta}\cdot\Delta\delta\boldsymbol{n}\;.\end{aligned} \tag{110}$$

Due to (28) and (30) we find

$$\boldsymbol{a}_{\alpha,\beta}\cdot\Delta\delta\boldsymbol{n} = \delta\boldsymbol{a}_\gamma\cdot\big(\Gamma^\gamma_{\alpha\beta}\,\boldsymbol{a}^\delta\otimes\boldsymbol{n} + \Gamma^\delta_{\alpha\beta}\,\boldsymbol{n}\otimes\boldsymbol{a}^\gamma - a^{\gamma\delta}\,b_{\alpha\beta}\,\boldsymbol{n}\otimes\boldsymbol{n}\big)\,\Delta\boldsymbol{a}_\delta\;, \tag{111}$$

which is symmetric w.r.t. variation and linearization. Inserting (111) into (110) and using (28), then gives

$$\begin{aligned}\Delta\delta b_{\alpha\beta} \;=\; & -\,\delta\boldsymbol{a}_\gamma\cdot(\boldsymbol{n}\otimes\boldsymbol{a}^\gamma)\,\Delta\boldsymbol{a}_{\alpha,\beta} - \delta\boldsymbol{a}_{\alpha,\beta}\cdot(\boldsymbol{a}^\gamma\otimes\boldsymbol{n})\,\Delta\boldsymbol{a}_\gamma \\ & +\;\delta\boldsymbol{a}_\gamma\cdot\big(\Gamma^\gamma_{\alpha\beta}\,\boldsymbol{a}^\delta\otimes\boldsymbol{n} + \Gamma^\delta_{\alpha\beta}\,\boldsymbol{n}\otimes\boldsymbol{a}^\gamma - a^{\gamma\delta}\,b_{\alpha\beta}\,\boldsymbol{n}\otimes\boldsymbol{n}\big)\,\Delta\boldsymbol{a}_\delta\;.\end{aligned} \tag{112}$$

4.5 Stability considerations

In order to assess the stability of the hyperelastic material models introduced above, we investigate the following stability condition: The material model is stable if there exists an $\epsilon > 0$ such that

$$g_{\alpha\beta}\, \hat{c}^{\alpha\beta\gamma\delta}\, g_{\gamma\delta} \geq \epsilon\, \|\boldsymbol{g}\|^2 \tag{113}$$

for any symmetric tensor $\boldsymbol{g} = g_{\alpha\beta}\, \boldsymbol{a}^\alpha \otimes \boldsymbol{a}^\beta$, see Marsden and Hughes (1994), p. 258. We will investigate this criteria for the membrane part ($\hat{c}^{\alpha\beta\gamma\delta} = c^{\alpha\beta\gamma\delta}$) and the bending part ($\hat{c}^{\alpha\beta\gamma\delta} = f^{\alpha\beta\gamma\delta}$) considering the examples in Sec. 5. In turns out that for those, the tangent components can all be written in the format

$$\hat{c}^{\alpha\beta\gamma\delta} = \hat{c}_{aa}\, a^{\alpha\beta}\, a^{\gamma\delta} + \hat{c}_a\, a^{\alpha\beta\gamma\delta} + \hat{c}_{ab}\, a^{\alpha\beta}\, b^{\gamma\delta} + \hat{c}_{ba}\, b^{\alpha\beta}\, a^{\gamma\delta} + \hat{c}_{bb}\, b^{\alpha\beta}\, b^{\gamma\delta}\ , \quad \hat{c} = c,\, d,\, e,\, f \tag{114}$$

for suitable definitions of coefficients $\hat{c}_{aa}$, $\hat{c}_a$, $\hat{c}_{ab}$, $\hat{c}_{ba}$ and $\hat{c}_{bb}$. If $\boldsymbol{g}$ is symmetric then its eigenvalues, following from $\det[g^\alpha_\beta - \lambda\, \delta^\alpha_\beta] = 0$, must be real, which implies that $g^1_2\, g^2_1 \geq 0$. For the special case $\hat{c}_{ab} = \hat{c}_{ba} = \hat{c}_{bb} = 0$, we then find

$$g_{\alpha\beta}\, \hat{c}^{\alpha\beta\gamma\delta}\, g_{\gamma\delta} = \hat{c}_{aa}\, (g^\alpha_\alpha)^2 - \hat{c}_a\, g^\alpha_\beta\, g^\beta_\alpha\ , \tag{115}$$

which further yields

$$g_{\alpha\beta}\, \hat{c}^{\alpha\beta\gamma\delta}\, g_{\gamma\delta} = \big(\hat{c}_{aa} - \hat{c}_a/2\big)\, (g^\alpha_\alpha)^2 - \hat{c}_a \left(\tfrac{1}{2}(g^1_1 - g^2_2)^2 + 2\, g^1_2\, g^2_1\right) . \tag{116}$$

Condition (113) is thus satisfied if both

$$2\hat{c}_{aa} - \hat{c}_a > 0 \quad \& \quad \hat{c}_a < 0\ . \tag{117}$$

Note, that criterion (113) is addressing material stability and does not say anything about structural instabilities, like buckling and wrinkling.

5 Stored energy functions for membranes and shells

This section presents several examples for hyperelastic shell models considering both solids and liquids. Specific models for area-incompressible shells are also considered. Eqs. (32), (34), (39), (42), (46), (91), (99) and (106) are then used to determine the stress and tangent components. From this the stability is then assessed. The first two sections examine pure membranes, for which W is only a function of $a_{\alpha\beta}$. The remaining two section deal with the bending part.

5.1 Membrane energy: solids

We begin by examining solid membranes. Two-cases are considered: Unconstrained membranes and area-constrained membranes.

5.1.1 Unconstrained solid membranes

A general model for unconstrained membranes is given by the stored surface energy density (per reference surface)

$$W = \frac{\Lambda}{4}(J^2 - 1 - 2\ln J) + \frac{\mu}{2}(I_1 - 2 - 2\ln J)\ . \tag{118}$$

This expression is very similar to the 3D Neo-Hookean model (Wriggers, 2008). By design, we have $W = 0$ if $J = 1$ and $a_{\alpha\beta} = A_{\alpha\beta}$. From (118) we find

$$\sigma^{\alpha\beta} = \frac{1}{J}\Big[\frac{\Lambda}{2}\big(J^2 - 1\big)a^{\alpha\beta} + \mu\big(A^{\alpha\beta} - a^{\alpha\beta}\big)\Big] \,, \tag{119}$$

while $M^{\alpha\beta} = 0$. We further find

$$c^{\alpha\beta\gamma\delta} = \Lambda\, J^2\, a^{\alpha\beta}\, a^{\gamma\delta} + \big(\Lambda\,(J^2 - 1) - 2\mu\big)\, a^{\alpha\beta\gamma\delta} \,, \tag{120}$$

while $d^{\alpha\beta\gamma\delta} = e^{\alpha\beta\gamma\delta} = f^{\alpha\beta\gamma\delta} = 0$. According to format (114),

$$\begin{aligned} c_{aa} &= \Lambda\, J^2 \,, \\ c_a &= \Lambda\,(J^2 - 1) - 2\mu \,, \end{aligned} \tag{121}$$

while all other $\hat{c}$ vanish. According to (117) this model is stable if parameters Λ and μ satisfy

$$K := \Lambda + \mu > 0 \quad \& \quad \mu > K\frac{J^2 - 1}{J^2 + 1} \,. \tag{122}$$

If $J > 1$, this is satisfied even for $\Lambda = 0$, which is the case considered in Sauer et al. (2014). The parameters K and μ are the in-plane bulk and shear moduli of the membrane.

Remark: The bulk and shear contributions can be separated if we consider the formulation

$$W = \frac{K}{4}(J^2 - 1 - 2\ln J) + \frac{\mu}{2}(\hat{I}_1 - 2) \,, \tag{123}$$

where $\hat{I}_1 = I_1/J$. Here the first term captures purely dilatoric deformation, while the second part captures purely deviatoric deformation. The formulation is analogous to the 3D case described for example in Wriggers (2008). It is considered further in Sauer et al. (2016).

5.1.2 Area-constrained solid membranes

Area-incompressible membranes observe the constraint

$$g = 1 - J = 0 \,. \tag{124}$$

This is included in the energy density function via the Lagrange multiplier method. Considering the formulation from Sec. 5.1.1 (with $J = 1$), we thus have

$$W = \frac{\mu}{2}(I_1 - 2) + q\, g \,, \tag{125}$$

such that

$$\tau^{\alpha\beta} = \mu\, A^{\alpha\beta} - qJ\, a^{\alpha\beta} \,, \tag{126}$$

and

$$c^{\alpha\beta\gamma\delta} = -qJ\, a^{\alpha\beta} a^{\gamma\delta} - 2qJ\, a^{\alpha\beta\gamma\delta} \,. \tag{127}$$

Criteria (117) are thus satisfied if $q > 0$, which is the case for the example in Sec. 7.2. Since $J = 1$, the Cauchy stress is now given by

$$\sigma^{\alpha\beta} = \mu\, A^{\alpha\beta} - q\, a^{\alpha\beta} \,. \tag{128}$$

5.2 Membrane energy: liquids

Next we examine liquid membranes under hydrostatic conditions, where no shear resistance is present. Three different models are considered.

5.2.1 Constant surface tension

The first model considers constant surface tension within the membrane, as it is often assumed for liquids. In that case the surface energy (per current area) is also constant and equal to the surface tension. Denoting the surface tension by $\gamma > 0$, the surface energy per reference area becomes

$$W = \gamma J \;, \tag{129}$$

and it follows that

$$\sigma^{\alpha\beta} = \gamma \, a^{\alpha\beta} \tag{130}$$

and

$$c^{\alpha\beta\gamma\delta} = \gamma \, J \left(a^{\alpha\beta} \, a^{\gamma\delta} + 2a^{\alpha\beta\gamma\delta} \right) \,. \tag{131}$$

According to format (114), we now have

$$\begin{aligned} c_{aa} &= \gamma \, J \;, \\ c_{a} &= 2\gamma \, J \;, \end{aligned} \tag{132}$$

while all other $\hat{c}$ vanish. This does not satisfy stability conditions (117). Liquid membranes therefore need to be stabilized in quasi-static computations, e.g. by the scheme proposed in Sauer (2014).
According to model (129), the surface area can increase without restriction during deformation. This is plausible for liquid membranes that bound liquids, since in that case the membrane can recruit surface molecules from the bulk during stretching. Likewise free films may be able to recruit further molecules through the boundary. If the liquid membrane is a free film, that cannot recruit molecules from the bulk or the boundary, it is more realistic to consider one of the following two models.

5.2.2 Area-compressible liquid membranes

If the liquid membrane is compressible, it can be modeled by Eq. (123) with $\mu = 0$. If the compressibility is only small the alternative

$$W = \frac{K}{2} g^2 \tag{133}$$

can be considered. Also here, $K > 0$ is the bulk modulus. It follows that

$$\sigma^{\alpha\beta} = K \, (J - 1) \, a^{\alpha\beta} \tag{134}$$

and

$$c^{\alpha\beta\gamma\delta} = KJ \, (2J - 1) \, a^{\alpha\beta} a^{\gamma\delta} + 2KJ \, (J - 1) \, a^{\alpha\beta\gamma\delta} \;. \tag{135}$$

This does not satisfy criteria (117) for $J \lesssim 1$. Essentially shear stiffness is missing.

5.2.3 Area-incompressible liquid membranes

Area-incompressible membranes are governed by

$$W = q\,g\;, \tag{136}$$

where g is given by (124). It follows that

$$\sigma^{\alpha\beta} = -q\,a^{\alpha\beta}\;. \tag{137}$$

The material tangent is identical to the one for area-incompressible solids in Eq. (127) above. Stability again requires $q > 0$. For liquids the stress at the boundary is usually tensile (e.g. equal to the surface tension $\gamma > 0$), such that $q = -\gamma < 0$. Model (136) is therefore also unstable.

5.3 Bending energy: liquids and solids

We now examine bending energies that have been proposed for liquid shells, but are in principle also applicable to solid shells. By liquid shells we understand shells that behave fluid-like in plane and solid-like out of plane. So the distinction really lies in the membrane part (discussed above) rather than the bending part. However, there are bending models that are unsuitable for liquids shells, as they also affect the in-plane response. Those are then discussed in Sec. 5.4. Here we discuss classical models developed for unconstrained liquid shells and area-constrained liquid shells.

5.3.1 Unconstrained shells

A popular bending model used to study liquid shells is the bending energy of Helfrich (1973)

$$w := k\,(H - H_0)^2 + \bar{k}\,\kappa \tag{138}$$

per current surface area. Here, H_0 denotes the so-called spontaneous curvature which is important to model for example the effect of proteins on lipid bilayers; k and $\bar{k}$ are material constants. Per reference area we then have

$$W = J\big(k\,(H - H_0)^2 + \bar{k}\,\kappa\big)\;. \tag{139}$$

Defining $\Delta H := H - H_0$, we now find

$$\tau^{\alpha\beta} = J\,\big(k\,\Delta H^2 - \bar{k}\,\kappa\big)\,a^{\alpha\beta} - 2k\,J\,\Delta H\,b^{\alpha\beta}\;, \tag{140}$$

$$M_0^{\alpha\beta} = J\,\big(k\,\Delta H + 2\bar{k}\,H\big)\,a^{\alpha\beta} - \bar{k}\,J\,b^{\alpha\beta}\;, \tag{141}$$

and further, using (45),

$$\begin{aligned} c^{\alpha\beta\gamma\delta} &= c_{aa}\,a^{\alpha\beta}\,a^{\gamma\delta} + c_a\,a^{\alpha\beta\gamma\delta} + c_{bb}\,b^{\alpha\beta}\,b^{\gamma\delta} + c_{ab}\,\big(a^{\alpha\beta}\,b^{\gamma\delta} + b^{\alpha\beta}\,a^{\gamma\delta}\big)\;, \\ d^{\alpha\beta\gamma\delta} &= d_{aa}\,a^{\alpha\beta}\,a^{\gamma\delta} + d_a\,a^{\alpha\beta\gamma\delta} + d_{ab}\,a^{\alpha\beta}\,b^{\gamma\delta} + d_{ba}\,b^{\alpha\beta}\,a^{\gamma\delta} = e^{\gamma\delta\alpha\beta}\;, \\ f^{\alpha\beta\gamma\delta} &= f_{aa}\,a^{\alpha\beta}\,a^{\gamma\delta} + f_a\,a^{\alpha\beta\gamma\delta}\;, \end{aligned} \tag{142}$$

with

$$\begin{aligned}
c_{aa} &= J\left(k\,\Delta H\,(\Delta H - 8H) + \bar{k}\,\kappa\right)\,, \\
c_{a} &= 2J\big(k\,\Delta H\,(\Delta H - 4H) - \bar{k}\,\kappa\big)\,, \\
c_{bb} &= 2k\,J\,, \\
c_{ab} &= c_{ba} = 2k\,J\,\Delta H\,, \\
d_{aa} &= J\left(k\,\Delta H - 2\bar{k}\,H\right)\,, \\
d_{a} &= 2J\,k\,\Delta H\,, \\
d_{ab} &= J\,\bar{k}\,, \\
d_{ba} &= -J\,k\,, \\
f_{aa} &= J\,(k/2 + \bar{k})\,, \\
f_{a} &= J\,\bar{k}\,.
\end{aligned} \tag{143}$$

As can be seen, this bending model generates membrane stresses. As was mentioned earlier for thin shells (see Sec. 3.3), those stresses are higher order contributions compared to the stresses originating from the membrane energies in Sec. 5.2. We will therefore assess the stability of the Helfrich model by only examining the bending response characterized by the tangent $f^{\alpha\beta\gamma\delta}$. According to (117), it is easy to see that the Helfrich model is only stable if

$$-k < \bar{k} < 0\,. \tag{144}$$

A special case of the Helfrich model is the bending model of Canham and Rand, initially proposed for red blood cells (Canham, 1970). It can be expressed as

$$W = J\,w\,, \quad w := \frac{c}{2}\left(\kappa_1^2 + \kappa_2^2\right)\,. \tag{145}$$

Alternatively, w can also be written as $w = c\,b^\alpha_\beta\,b^\beta_\alpha/2$ or $w = c\,(2H^2 - \kappa)$, so that it follows from the Helfrich model with $k = 2c$, $\bar{k} = -c$ and $H_0 = 0$. Since this satisfies (144), the model is stable in bending. In particular we get for the Caham model,

$$\sigma^{\alpha\beta} = c\,(2H^2 + \kappa)\,a^{\alpha\beta} - 4c\,H\,b^{\alpha\beta} \tag{146}$$

and

$$M^{\alpha\beta} = c\,b^{\alpha\beta}\,. \tag{147}$$

If we ignore high order dependencies on the deformation, we arrive at the classical plate equations $\sigma^{\alpha\beta} = 0$ and $M^{\alpha\beta} = c\,b^{\alpha\beta}$.

5.3.2 Area-constrained shells

The Helfrich model can be adapted to area-incompressible shells. We then have

$$W = k\,\Delta H^2 + \bar{k}\,\kappa\,. \tag{148}$$

which can then be combined with Eq. (136). From (148) follows

$$\begin{aligned}
\tau^{\alpha\beta} &= -2k\,\Delta H\,b^{\alpha\beta} - 2\bar{k}\,\kappa\,a^{\alpha\beta} &&= \sigma^{\alpha\beta}\,, \\
M_0^{\alpha\beta} &= \big(k\,\Delta H + 2\bar{k}\,H\big)\,a^{\alpha\beta} - \bar{k}\,b^{\alpha\beta} &&= M^{\alpha\beta}\,,
\end{aligned} \tag{149}$$

from which we now find the tangent coefficients

$$\begin{aligned}
c_{aa} &= -8k\,H\,\Delta H + 4\bar{k}\,\kappa\ , \\
c_{a} &= -8k\,H\,\Delta H - 4\bar{k}\,\kappa\ , \\
c_{bb} &= 2k\ , \\
c_{ab} &= c_{ba} = 4k\,\Delta H\ , \\
d_{aa} &= -4\bar{k}\,H, \\
d_{a} &= 2k\,\Delta H\ , \\
d_{ab} &= 2\,\bar{k}\ , \\
d_{ba} &= -k\ , \\
f_{aa} &= \big(k/2 + \bar{k}\big)\ , \\
f_{a} &= \bar{k}\ .
\end{aligned} \tag{150}$$

Bending stability again follows if condition (144) is met.

For the Canham special case we now have

$$\sigma^{\alpha\beta} = 2c\,(\kappa a^{\alpha\beta} - 2\,H\,b^{\alpha\beta}) \tag{151}$$

and

$$M^{\alpha\beta} = c\,b^{\alpha\beta}\ . \tag{152}$$

5.4 Bending energy: solids

We finally examine some classical bending models that have been developed for solid shells. The following two models are examined: the Koiter model and the classical 3D modeling approach.

5.4.1 Linear elastic shells: The Koiter model

A very simple shell model is given by the model of Koiter, e.g. see Ciarlet (2005). It can be expressed as

$$W = \frac{1}{8}\big(a_{\alpha\beta} - A_{\alpha\beta}\big)\,c^{\alpha\beta\gamma\delta}\,\big(a_{\gamma\delta} - A_{\gamma\delta}\big) + \frac{1}{2}\big(b_{\alpha\beta} - B_{\alpha\beta}\big)\,f^{\alpha\beta\gamma\delta}\,\big(b_{\gamma\delta} - B_{\gamma\delta}\big) \tag{153}$$

where $B^{\alpha\beta}$ denotes the curvature in the initial configuration, and where

$$\begin{aligned}
c^{\alpha\beta\gamma\delta} &= \Lambda\,A^{\alpha\beta}\,A^{\gamma\delta} + \mu\big(A^{\alpha\gamma}\,A^{\beta\delta} + A^{\alpha\delta}\,A^{\beta\gamma}\big)\ , \\
f^{\alpha\beta\gamma\delta} &= \frac{T^2}{12}\,c^{\alpha\beta\gamma\delta}\ ,
\end{aligned} \tag{154}$$

are now constants. Here T is the initial shell thickness. It follows that

$$\begin{aligned}
\tau^{\alpha\beta} &= c^{\alpha\beta\gamma\delta}\,\big(a_{\gamma\delta} - A_{\gamma\delta}\big)/2\ , \\
M_0^{\alpha\beta} &= f^{\alpha\beta\gamma\delta}\,\big(b_{\gamma\delta} - B_{\gamma\delta}\big)\ .
\end{aligned} \tag{155}$$

From Eq. (106) further follows that $d^{\alpha\beta\gamma\delta} = e^{\alpha\beta\gamma\delta} = 0$. The Koiter shell model is analogous to the St. Venant-Kirchhoff model in classical continuum mechanics. It yields linear material relations for $\tau^{\alpha\beta}(a_{\gamma\delta})$ and $M_0^{\alpha\beta}(b_{\gamma\delta})$, while geometrical nonlinearities are still captured. If desired, the membrane part within (153) can be easily replaced by one of the nonlinear models of Sec. 5.1.

5.4.2 Bending models derived from 3D constitutive models

Computational shell models are often based on classical three-dimensional constitutive models of the form $\tilde{W} = \tilde{W}(\tilde{\boldsymbol{C}})$, where $\tilde{\boldsymbol{C}}$ is the Cauchy-Green tensor for 3D continua. Considering the special kinematics of shells, we can then write $\tilde{\boldsymbol{C}} = \tilde{\boldsymbol{C}}(a^{\alpha\beta}, b^{\alpha\beta})$, such that $W = W(a^{\alpha\beta}, b^{\alpha\beta})$ can be extracted. Specifically we have

$$\tilde{\boldsymbol{C}} = C_{\alpha\beta}\,\boldsymbol{G}^\alpha \otimes \boldsymbol{G}^\beta + C_{\alpha 3}\left(\boldsymbol{G}^\alpha \otimes \boldsymbol{N} + \boldsymbol{N} \otimes \boldsymbol{G}^\alpha\right) + C_{33}\,\boldsymbol{N} \otimes \boldsymbol{N}\;, \tag{156}$$

where $\boldsymbol{G}^\alpha = G^{\alpha\beta}\,\boldsymbol{G}_\beta$ and $[G^{\alpha\beta}] = [G_{\alpha\beta}]^{-1}$ with $G_{\alpha\beta} = \boldsymbol{G}_\alpha \cdot \boldsymbol{G}_\beta$. $\{\boldsymbol{G}_1, \boldsymbol{G}_2, \boldsymbol{N}\}$ is the basis used to describe the initial shell geometry, e.g. see Wriggers (2008). $\boldsymbol{G}_\alpha$ acounts for the surface stretch due to the initial shell curvature, i.e.

$$\boldsymbol{G}_\alpha = \left(\delta^\beta_\alpha - \xi\, B^\beta_\alpha\right) \boldsymbol{A}_\beta\;, \tag{157}$$

where $\xi \in [-T/2,\, T/2]$ is the thickness coordinates of the shell. For the Kirchhoff-Love shell we have

$$C_{\alpha\beta} = a_{\alpha\beta} + 2\xi\, b_{\alpha\beta} + \xi^2\, b^\gamma_\alpha\, b_{\gamma\beta}\;, \tag{158}$$

while $C_{\alpha 3} = 0$ and $C_{33} = 1$. The surface energy of the shell then follows from the thickness integration

$$W = \int_{-T/2}^{T/2} \tilde{W}\,\mathrm{d}\xi\;. \tag{159}$$

From this all the stress, moment and tangent components follow.
The advantages of this formulation is that any $\tilde{W}$ can be used, and that it is not restricted to Kirchhoff-Love kinematics. On the downside, numerical quadrature has to be used generally in order to evalute the thickness integration coming from (159). In the case of pure membranes, the dependency on ξ is neglected, and the integration can be carried out analytically (Sauer, 2016).

6 Weak form

This section presents the weak form of the Kirchhoff-Love shell, and discusses its linearization and decomposition into in-plane and out-of-plane contributions. Unconstrained and constrained shells are considered.

6.1 Unconstrained system

The weak form of equilibrium equation (61) can be derived analogously to the mechanical power balance in Sec. 4.1 by simply replacing the velocity $\boldsymbol{v}$ with the admissible variation $\delta\boldsymbol{x} \in \mathcal{V}$. Immediatly we obtain

$$G_{\mathrm{in}} + G_{\mathrm{int}} - G_{\mathrm{ext}} = 0 \quad \forall\,\delta\boldsymbol{x} \in \mathcal{V}\;, \tag{160}$$

with

$$\begin{aligned} G_{\mathrm{in}} &= \int_{\mathcal{S}_0} \delta\boldsymbol{x}\cdot\rho_0\,\dot{\boldsymbol{v}}\,\mathrm{d}A\;, \\ G_{\mathrm{int}} &= \int_{\mathcal{S}_0} \delta_{\mathrm{x}} W\,\mathrm{d}A = \int_{\mathcal{S}_0} \frac{1}{2}\,\delta a_{\alpha\beta}\,\tau^{\alpha\beta}\,\mathrm{d}A + \int_{\mathcal{S}_0} \delta b_{\alpha\beta}\,M_0^{\alpha\beta}\,\mathrm{d}A\;, \\ G_{\mathrm{ext}} &= \int_{\mathcal{S}} \delta\boldsymbol{x}\cdot\boldsymbol{f}\,\mathrm{d}a + \int_{\partial\mathcal{S}} \delta\boldsymbol{x}\cdot\boldsymbol{T}\,\mathrm{d}s + \int_{\partial\mathcal{S}} \delta\boldsymbol{n}\cdot\boldsymbol{M}\,\mathrm{d}s\;, \end{aligned} \tag{161}$$

according to Eqs. (87)–(90) and (82). Due to Eq. (102), $G_{\rm int}$ can also be obtained as the variation of

$$\Pi_{\rm int} = \int_{\mathcal{S}_0} W\,\mathrm{d}A \tag{162}$$

w.r.t. $\boldsymbol{x}$, i.e. $G_{\rm int} = \delta_{\rm x}\Pi_{\rm int}$. Thus, if $G_{\rm ext}$ is also derivable from a potential, the static weak form $G_{\rm int} - G_{\rm ext} = 0\ \forall\,\delta\boldsymbol{x} \in \mathcal{V}$ is the result of the principle of stationary potential energy.

6.2 Constrained system

For the constrained problem, we need to include the constraint $g = 0$. The weak form of that is simply

$$G_{\rm g} = \int_{\mathcal{S}_0} \delta q\, g\,\mathrm{d}A = 0 \quad \forall\,\delta q \in \mathcal{Q}~, \tag{163}$$

where $\delta q \in \mathcal{Q}$ is a suitably chosen variation of the Langange multiplier q. The weak form problem statement is then given by solving the two equations

$$\begin{aligned} G_{\rm in} + G_{\rm int} - G_{\rm ext} &= 0 \quad \forall\,\delta\boldsymbol{x} \in \mathcal{V}~, \\ G_{\rm g} &= 0 \quad \forall\,\delta q \in \mathcal{Q}~, \end{aligned} \tag{164}$$

for $\boldsymbol{x}$ and q. Due to Eq. (100), we can find $G_{\rm int} + G_{\rm g} = \delta\Pi_{\rm int}$, such that the static version of weak form (164), for suitable $G_{\rm ext}$, is still the result of the principle of stationary potential energy.

6.3 On the application of boundary moments and tractions

Based on definition (76) from Sec. 3.4, we find

$$\delta\boldsymbol{x}\cdot\boldsymbol{T} = \delta\boldsymbol{x}\cdot\boldsymbol{t} + \delta\boldsymbol{n}\cdot m_\nu\,\boldsymbol{\tau} + (\delta\boldsymbol{x}\cdot m_\nu\,\boldsymbol{n})'~, \tag{165}$$

by using Eq. (28) and

$$\delta\boldsymbol{x}' = \frac{\partial\delta\boldsymbol{x}}{\partial\xi^\alpha}\frac{\partial\xi^\alpha}{\partial s} = \delta\boldsymbol{a}_\alpha\,\tau^\alpha = \delta\boldsymbol{a}_\alpha\,(\boldsymbol{a}^\alpha\cdot\boldsymbol{\tau})~. \tag{166}$$

With the help of Eq. (58), we thus obtain

$$\delta\boldsymbol{x}\cdot\boldsymbol{T} + \delta\boldsymbol{n}\cdot\boldsymbol{M} = \delta\boldsymbol{x}\cdot\boldsymbol{t} + \delta\boldsymbol{n}\cdot m_\tau\,\boldsymbol{\nu} + (\delta\boldsymbol{x}\cdot m_\nu\,\boldsymbol{n})'~, \tag{167}$$

such that

$$\int_{\partial\mathcal{S}} \big(\delta\boldsymbol{x}\cdot\boldsymbol{T} + \delta\boldsymbol{n}\cdot\boldsymbol{M}\big)\,\mathrm{d}s = \int_{\partial\mathcal{S}} \big(\delta\boldsymbol{x}\cdot\boldsymbol{t} + \delta\boldsymbol{n}\cdot m_\tau\,\boldsymbol{\nu}\big)\,\mathrm{d}s + \big[\delta\boldsymbol{x}\cdot m_\nu\,\boldsymbol{n}\big]~, \tag{168}$$

where the last term denotes the virtual work of the point loads $m_\nu\,\boldsymbol{n}$ that are present at corners on the boundary $\partial\mathcal{S}$. At Dirichlet boundaries usually $\delta\boldsymbol{x} = 0$. If, at Neumann boundaries, $m_\nu = 0$, i.e. only $\boldsymbol{m} = m_\tau\boldsymbol{\tau}$ is applied, the last term in Eq. (168) vanishes.

6.4 Linearization

For the linearization of the weak form, let us consider the static case of (164), written in the combined form

$$\delta\Pi_{\mathrm{int}} - G_{\mathrm{ext}} = 0 \quad \forall\, \delta\boldsymbol{x} \in \mathcal{V} \;\&\; \delta q \in \mathcal{Q}\;, \tag{169}$$

where $\delta\Pi_{\mathrm{int}} = G_{\mathrm{int}} + G_{\mathrm{g}}$. Linearizing the internal virtual work gives, according to (103),

$$\Delta\delta\Pi_{\mathrm{int}} = \int_{\mathcal{S}_0} \Delta_{\boldsymbol{x}}\delta_{\boldsymbol{x}} W \,\mathrm{d}A + \int_{\mathcal{S}_0} \delta g\,\Delta q \,\mathrm{d}A + \int_{\mathcal{S}_0} \delta q\,\Delta g\,\mathrm{d}A\;, \tag{170}$$

where $\Delta_{\boldsymbol{x}}\delta_{\boldsymbol{x}} W$ is given by (105). In order to linearize G_{ext} we consider dead loading for $\boldsymbol{f}$, $\boldsymbol{t}$ and $\boldsymbol{M}$. The case of live pressure loading is given in Sauer et al. (2014). For dead $\boldsymbol{M}$, we must have

$$m_\tau\,\mathrm{d}s = m_\tau^0\,\mathrm{d}S = \mathrm{const.} \tag{171}$$

The linearization of G_{ext} according to (161.3) and (168) thus only yields

$$\Delta G_{\mathrm{ext}} = \int_{\partial\mathcal{S}} \Delta\delta\boldsymbol{n}\cdot m_\tau\,\boldsymbol{\nu}\,\mathrm{d}s + \int_{\partial\mathcal{S}} \delta\boldsymbol{n}\cdot m_\tau\,\Delta\boldsymbol{\nu}\,\mathrm{d}s\;. \tag{172}$$

From (28) and (30) we find

$$\Delta\delta\boldsymbol{n} = (\delta\boldsymbol{a}_\alpha\cdot\boldsymbol{n})(\boldsymbol{n}\cdot\Delta\boldsymbol{a}_\beta)\,a^{\alpha\beta}\,\boldsymbol{n} + (\delta\boldsymbol{a}_\alpha\cdot\boldsymbol{n})(\boldsymbol{a}^\alpha\cdot\Delta\boldsymbol{a}_\beta)\,\boldsymbol{a}^\beta + (\delta\boldsymbol{a}_\alpha\cdot\boldsymbol{a}^\beta)(\boldsymbol{n}\cdot\Delta\boldsymbol{a}_\beta)\,\boldsymbol{a}^\alpha\;, \tag{173}$$

while in Sauer (2014), Appendix A.4, we have showed that for $\boldsymbol{\nu} = \boldsymbol{\tau}\times\boldsymbol{n}$ we have

$$\Delta\boldsymbol{\nu} = -(\boldsymbol{\tau}\otimes\boldsymbol{\nu})\,\Delta\boldsymbol{\tau} - (\boldsymbol{n}\otimes\boldsymbol{\nu})\,\Delta\boldsymbol{n}\;, \tag{174}$$

where $\Delta\boldsymbol{\tau} = \Delta\boldsymbol{x}' = \partial\Delta\boldsymbol{x}/\partial s$. For dead loading, $\Delta\boldsymbol{\tau} = \boldsymbol{0}$. Further, $\delta\boldsymbol{n}\cdot\boldsymbol{n} = 0$. The second term in (172) thus vanishes and we are only left with

$$\Delta G_{\mathrm{ext}} = \int_{\partial\mathcal{S}} m_\tau\,\delta\boldsymbol{a}_\alpha\cdot\big(\nu^\beta\,\boldsymbol{n}\otimes\boldsymbol{a}^\alpha + \nu^\alpha\,\boldsymbol{a}^\beta\otimes\boldsymbol{n}\big)\,\Delta\boldsymbol{a}_\beta\,\mathrm{d}s\;, \tag{175}$$

which is symmetric w.r.t. variation and linearization, as it should be for dead loading.

6.5 Decomposition

The weak form can be decomposed into in-plane and out-of-plane contributions, which is discussed here. This may be interesting for liquids, when hydrostatic stabilization approaches are developed based on the in-plane weak form (Sauer et al., 2014; Sauer, 2014).
We first consider the terms in G_{int}. Let $\delta\boldsymbol{x} = \boldsymbol{w} = w_\alpha\,\boldsymbol{a}^\alpha + w\,\boldsymbol{n}$, such that $\delta\boldsymbol{a}_\alpha = \boldsymbol{w}_{,\alpha} = \boldsymbol{w}_{;\alpha}$. Since

$$w_{\alpha;\beta} = (\boldsymbol{w}\cdot\boldsymbol{a}_\alpha)_{;\beta} = \boldsymbol{w}_{;\beta}\cdot\boldsymbol{a}_\alpha + w\,b_{\alpha\beta}\;, \tag{176}$$

we find

$$\delta a_{\alpha\beta} = w_{\alpha;\beta} + w_{\beta;\alpha} - 2w\,b_{\alpha\beta}\;. \tag{177}$$

For symmetric $\sigma^{\alpha\beta}$, we thus find the following decomposition,

$$\tfrac{1}{2}\sigma^{\alpha\beta}\,\delta a_{\alpha\beta} = w_{\alpha;\beta}\,\sigma^{\alpha\beta} - w\,b_{\alpha\beta}\,\sigma^{\alpha\beta}\;. \tag{178}$$

Taking one and two derivatives of $\boldsymbol{w}$ and contracting with $\boldsymbol{n}$ gives

$$\begin{aligned} \boldsymbol{w}_{;\alpha}\cdot\boldsymbol{n} &= w_\beta\,b^\beta_\alpha + w_{;\alpha}\;,\\ \boldsymbol{w}_{;\alpha\beta}\cdot\boldsymbol{n} &= w_{\gamma;\alpha}\,b^\gamma_\beta + w_{\gamma;\beta}\,b^\gamma_\alpha + w_\gamma\,b^\gamma_{\alpha;\beta} + w_{;\alpha\beta} - w\,b_{\alpha\gamma}\,b^\gamma_\beta\;, \end{aligned} \tag{179}$$

such that we can express $\delta b_{\alpha\beta}$ according to (29) as

$$\delta b_{\alpha\beta} = (\boldsymbol{w}_{;\alpha\beta} - \Gamma^{\gamma}_{\alpha\beta}\,\boldsymbol{w}_{;\gamma}) \cdot \boldsymbol{n} \ . \tag{180}$$

For symmetric $M^{\alpha\beta}$, we thus obtain the decomposition

$$M^{\alpha\beta}\,\delta b_{\alpha\beta} = \Big[2w_{\gamma;\alpha}\,b^{\gamma}_{\beta} + w_{\gamma}\,\big(b^{\gamma}_{\alpha;\beta} - b^{\gamma}_{\delta}\,\Gamma^{\delta}_{\alpha\beta}\big)\Big]\,M^{\alpha\beta} + \Big[w_{;\alpha\beta} - w_{;\gamma}\,\Gamma^{\gamma}_{\alpha\beta} - w\,b_{\alpha\gamma}\,b^{\gamma}_{\beta}\Big]\,M^{\alpha\beta} \ . \tag{181}$$

From $b^{\gamma}_{\alpha} = a^{\gamma\delta}\,b_{\delta\alpha}$ it can be shown that

$$b^{\gamma}_{\alpha;\beta} = a^{\gamma\delta}\,b_{\delta\alpha;\beta} \ , \tag{182}$$

since $a^{\gamma\delta}{}_{;\beta} = 0$. Further

$$b_{\delta\alpha;\beta} = \boldsymbol{n} \cdot \boldsymbol{a}_{\delta;\alpha\beta} \ , \tag{183}$$

since $\boldsymbol{n}_{;\beta} \cdot \boldsymbol{a}_{\delta;\alpha} = 0$.

Next, let us consider the terms of G_{in} and G_{ext}. Writing $\dot{\boldsymbol{v}} = \boldsymbol{a} := a^{\alpha}\,\boldsymbol{a}_{\alpha} + a_{\mathrm{n}}\,\boldsymbol{n}$, $\boldsymbol{f} := f^{\alpha}\,\boldsymbol{a}_{\alpha} + p\,\boldsymbol{n}$ and $\boldsymbol{t} := t^{\alpha}\,\boldsymbol{a}_{\alpha} + t_{\mathrm{n}}\,\boldsymbol{n}$, we find

$$\begin{aligned} \delta\boldsymbol{x} \cdot \dot{\boldsymbol{v}} &= w_{\alpha}\,a^{\alpha} + w\,a_{\mathrm{n}} \ , \\ \delta\boldsymbol{x} \cdot \boldsymbol{f} &= w_{\alpha}\,f^{\alpha} + w\,p \ , \\ \delta\boldsymbol{x} \cdot \boldsymbol{t} &= w_{\alpha}\,t^{\alpha} + w\,t_{\mathrm{n}} \ . \end{aligned} \tag{184}$$

From (28) and (179) we further find

$$\delta\boldsymbol{n} \cdot m_{\tau}\,\boldsymbol{\nu} = -\big(w_{;\alpha} + b^{\beta}_{\alpha}\,w_{\beta}\big)\,\nu^{\alpha}\,m_{\tau} \ . \tag{185}$$

With the help of these equations and (168), the weak form of (160)–(161) can then be decomposed by alternatively setting $w = 0$ and $w_{\alpha} = 0$. We thus obtain the in-plane and out-of-plane weak forms

$$\begin{aligned} G_{\mathrm{ini}} + G_{\mathrm{inti}} + G_{\mathrm{exti}} &= 0 \quad \forall\, w_{\alpha} \in \mathcal{V}_{\alpha} \ , \\ G_{\mathrm{ino}} + G_{\mathrm{into}} + G_{\mathrm{exto}} &= 0 \quad \forall\, w \in \mathcal{V}_{\mathrm{n}} \ , \end{aligned} \tag{186}$$

with

$$\begin{aligned} G_{\mathrm{ini}} &= \int_{\mathcal{S}} w_{\alpha}\,\rho\,a^{\alpha}\,\mathrm{d}A \ , \\ G_{\mathrm{ino}} &= \int_{\mathcal{S}} w\,\rho\,a_{\mathrm{n}}\,\mathrm{d}A \ , \\ G_{\mathrm{inti}} &= \int_{\mathcal{S}} w_{\alpha;\beta}\,\sigma^{\alpha\beta}\,\mathrm{da} + \int_{\mathcal{S}} \Big[2w_{\gamma;\alpha}\,b^{\gamma}_{\beta} + w_{\gamma}\,\big(b^{\gamma}_{\alpha;\beta} - b^{\gamma}_{\delta}\,\Gamma^{\delta}_{\alpha\beta}\big)\Big]\,M^{\alpha\beta}\,\mathrm{da} \ , \\ G_{\mathrm{into}} &= -\int_{\mathcal{S}} w\,b_{\alpha\beta}\,\sigma^{\alpha\beta}\,\mathrm{da} + \int_{\mathcal{S}} \Big[w_{;\alpha\beta} - w_{;\gamma}\,\Gamma^{\gamma}_{\alpha\beta} - w\,b_{\alpha\gamma}\,b^{\gamma}_{\beta}\Big]\,M^{\alpha\beta}\,\mathrm{da} \ , \\ G_{\mathrm{exti}} &= \int_{\mathcal{S}} w_{\alpha}\,f^{\alpha}\,\mathrm{da} + \int_{\partial\mathcal{S}} w_{\alpha}\,t^{\alpha}\,\mathrm{ds} - \int_{\partial\mathcal{S}} w_{\alpha}\,b^{\alpha\beta}\,\nu_{\beta}\,m_{\tau}\,\mathrm{ds} \ , \\ G_{\mathrm{exto}} &= \int_{\mathcal{S}} w\,p\,\mathrm{da} + \int_{\partial\mathcal{S}} w\,t_{\mathrm{n}}\,\mathrm{ds} - \int_{\partial\mathcal{S}} w_{;\alpha}\,\nu^{\alpha}\,m_{\tau}\,\mathrm{ds} + [w\,m_{\nu}] \ . \end{aligned} \tag{187}$$

7 Analytical solution for pure bending

To illustrate the shell models from above let us consider a simple example. We consider a flat rectangular sheet with dimension $S \times L$, parameterized by the coordinates $\xi \in [0, S]$ and $\eta \in [0, L]$. The sheet is deformed into a curved sheet with dimension $s \times \ell$ by applying the

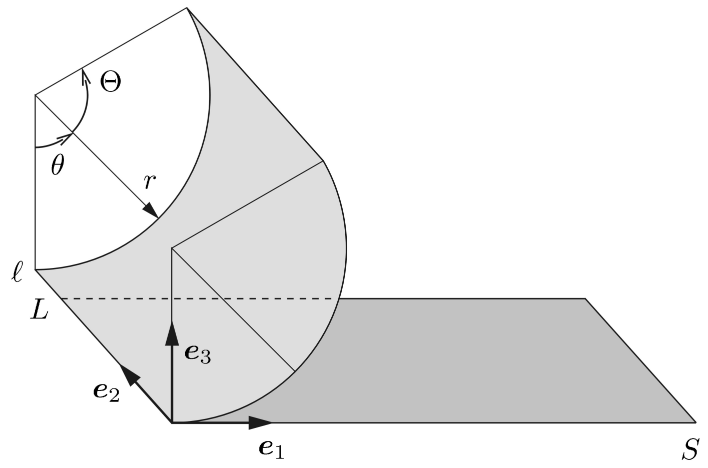

Figure 2: Pure bending example: Deformation of a flat sheet into a curved sheet with constant radius.

homogeneous curvature κ_1 and the homogeneous stretches $\lambda_1 = s/S$ and $\lambda_2 = \ell/L$ as is shown in Fig. 2. The deformed sheet thus forms a circular arc with radius $r = 1/\kappa_1$. The parameters S, L, κ_1, λ_1 and λ_2 are considered given, unless specified otherwise. According to the figure, the surface in its initial configuration can be described by

$$\boldsymbol{X}(\xi,\eta) = \xi\,\boldsymbol{e}_1 + \eta\,\boldsymbol{e}_2 \ , \tag{188}$$

while its current surface can be described by

$$\boldsymbol{x}(\xi,\eta) = r\sin\theta\,\boldsymbol{e}_1 + \lambda_2\,\eta\,\boldsymbol{e}_2 + r\,(1-\cos\theta)\,\boldsymbol{e}_3 \ , \tag{189}$$

with $\theta := \kappa_1\lambda_1\xi$ and $r := 1/\kappa_1$. The rotation at the end thus is $\Theta = \kappa_1\lambda_1 S$. From these relations we obtain the initial tangent vectors

$$\begin{aligned} \boldsymbol{A}_1 &= \frac{\partial\boldsymbol{X}}{\partial\xi} = \boldsymbol{e}_1 \ , \\ \boldsymbol{A}_2 &= \frac{\partial\boldsymbol{X}}{\partial\eta} = \boldsymbol{e}_2 \ , \end{aligned} \tag{190}$$

the current tangent vectors

$$\begin{aligned} \boldsymbol{a}_1 &= \frac{\partial\boldsymbol{x}}{\partial\xi} = \lambda_1\big(\cos\theta\,\boldsymbol{e}_1 + \sin\theta\,\boldsymbol{e}_3\big) \ , \\ \boldsymbol{a}_2 &= \frac{\partial\boldsymbol{x}}{\partial\eta} = \lambda_2\,\boldsymbol{e}_2 \ , \end{aligned} \tag{191}$$

and the current normal

$$\boldsymbol{n} = -\sin\theta\,\boldsymbol{e}_1 + \cos\theta\,\boldsymbol{e}_3 \ . \tag{192}$$

From these we find the kinematic quantities

$$[A_{\alpha\beta}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad [A^{\alpha\beta}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{193}$$

$$[a_{\alpha\beta}] = \begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix}, \quad [a^{\alpha\beta}] = \begin{bmatrix} \lambda_1^{-2} & 0 \\ 0 & \lambda_2^{-2} \end{bmatrix}, \quad J = \lambda_1\lambda_2 \ , \tag{194}$$

and

$$[b_{\alpha\beta}] = \begin{bmatrix} \kappa_1\lambda_1^2 & 0 \\ 0 & 0 \end{bmatrix}, \quad [b^\alpha_\beta] = \begin{bmatrix} \kappa_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad [b^{\alpha\beta}] = \begin{bmatrix} \kappa_1\lambda_1^{-2} & 0 \\ 0 & 0 \end{bmatrix}, \quad H = \frac{\kappa_1}{2} \ , \quad \kappa = 0 \ . \tag{195}$$

7.1 Unconstrained solid shell

Let us first consider the unconstrained solid models of Sec. 5.1.1 and 5.3.1. Substituting the above relations into Eq. (119) yields the membrane stress

$$[\sigma^{\alpha\beta}] = \frac{\Lambda}{2J}\big(J^2-1\big)\begin{bmatrix} \lambda_1^{-2} & 0 \\ 0 & \lambda_2^{-2} \end{bmatrix} + \frac{\mu}{J}\begin{bmatrix} 1-\lambda_1^{-2} & 0 \\ 0 & 1-\lambda_2^{-2} \end{bmatrix}, \tag{196}$$

due to stretches λ_1 and λ_2. Evaluating (146) and (147) yields the stress and bending moment due to κ_1,

$$[\sigma^{\alpha\beta}] = \frac{c}{2}\kappa_1^2\begin{bmatrix} -3\lambda_1^{-2} & 0 \\ 0 & \lambda_2^{-2} \end{bmatrix} \tag{197}$$

and

$$[M^{\alpha\beta}] = c\,\kappa_1\begin{bmatrix} \lambda_1^{-2} & 0 \\ 0 & 0 \end{bmatrix}, \tag{198}$$

such that

$$[b^\alpha_\gamma M^{\gamma\beta}] = c\,\kappa_1^2\begin{bmatrix} \lambda_1^{-2} & 0 \\ 0 & 0 \end{bmatrix}. \tag{199}$$

Now consider a cut at θ that is perpendicular to the normal

$$\boldsymbol{\nu} = \boldsymbol{a}_1/\lambda_1\ , \tag{200}$$

such that

$$\nu_1 = \boldsymbol{a}_1\cdot\boldsymbol{\nu} = \lambda_1 \quad \text{and} \quad \nu_2 = \boldsymbol{a}_2\cdot\boldsymbol{\nu} = 0\ . \tag{201}$$

The distributed bending moment acting on the cut is given by $M = M^{\alpha\beta}\nu_\alpha\nu_\beta$. For the present example we therefore find the simple linear relation

$$M = c\,\kappa_1\ , \tag{202}$$

between the prescribed curvature and the resulting bending moment.
Inserting (196)–(199) into (73) yields the in-plane stress components $N^{12} = N^{21} = 0$ and

$$\begin{aligned} N^{11} &= \frac{\Lambda}{2J}\big(J^2-1\big)\lambda_1^{-2} + \frac{\mu}{J}\big(1-\lambda_1^{-2}\big) - \frac{c}{2}\,\kappa_1^2\lambda_1^{-2}\ , \\ N^{22} &= \frac{\Lambda}{2J}\big(J^2-1\big)\lambda_2^{-2} + \frac{\mu}{J}\big(1-\lambda_2^{-2}\big) + \frac{c}{2}\,\kappa_1^2\lambda_2^{-2}\ , \end{aligned} \tag{203}$$

due to combined bending and stretching. Therefore we have $N^1_2 = N^2_1 = 0$ and

$$\begin{aligned} N^1_1 &= \frac{\Lambda}{2J}\big(J^2-1\big) + \frac{\mu}{J}\big(\lambda_1^2-1\big) - \frac{c}{2}\,\kappa_1^2\ , \\ N^2_2 &= \frac{\Lambda}{2J}\big(J^2-1\big) + \frac{\mu}{J}\big(\lambda_2^2-1\big) + \frac{c}{2}\,\kappa_1^2\ . \end{aligned} \tag{204}$$

As can be seen from the last term, the bending affects the in-plane membrane stress state. This influence may seem odd, but it is a high order effect since the bending stiffness c is usually proportional to the shell thickness cubed, while Λ and μ are only proportional to the thickness itself. The influence thus vanished for thinner and thinner shells.

To assess the solution further, the following two special cases are considered next:

1. Dirichlet case: Bending is induced by prescribing a rotation on the two edges such that the curvature is κ_1. Further, all edges are fixed such that $\lambda_1 = \lambda_2 = 1$. The forces at the boundary are thus $M = c\,\kappa_1$ and

$$\begin{aligned} N^1_1 &= -\frac{c}{2}\,\kappa_1^2 \,, \\ N^2_2 &= +\frac{c}{2}\,\kappa_1^2 \,. \end{aligned} \tag{205}$$

Since $N^1_1 \neq 0$ the surface pressure $p = N^1_1/r$ is required to equilibrate the structure.

2. Neumann case: Now the bending moment M is prescribed and the boundaries are free to move. The curvature now is $\kappa_1 = M/c$, while the stretches can be determined from (204). Since $N^1_1 = N^2_2 = 0$, we get

$$\begin{aligned} 0 &= \Lambda\,(J^2 - 1) + 2\mu\,(\lambda_1^2 - 1) - c\,J\,\kappa_1^2 \,, \\ 0 &= \Lambda\,(J^2 - 1) + 2\mu\,(\lambda_2^2 - 1) + c\,J\,\kappa_1^2 \,. \end{aligned} \tag{206}$$

Since $J = \lambda_1\lambda_2$, this is equivalent to

$$\begin{aligned} 0 &= \frac{\lambda_1}{\lambda_2} - \frac{\lambda_2}{\lambda_1} - \frac{c}{\mu}\kappa_1^2 \\ 0 &= \Lambda\,(\lambda_1^2\lambda_2^2 - 1) + \mu\,(\lambda_1^2 + \lambda_2^2 - 2) \,. \end{aligned} \tag{207}$$

Taking into account Eq. (202), Eq. (207) yields the physical solution

$$\lambda_2 = \lambda_1/a_0 \,, \quad a_0 := \frac{M^2}{2\mu c} + \sqrt{\left(\frac{M^2}{2\mu c}\right)^2 + 1} \tag{208}$$

and

$$\lambda_1 = \sqrt{-\bar\mu\,(a_0^2 + 1) + \sqrt{\bar\mu^2(a_0^2+1)^2 + a_0^2\,(4\,\bar\mu + 1)}} \,, \quad \bar\mu := \frac{\mu}{2\Lambda} \,. \tag{209}$$

7.2 Area-constrained solid shell

Next, we consider the area-constrained solid model considering the models in Secs. 5.1.2 and 5.3.2. Since now $J = 1$ we have $\lambda_2 = 1/\lambda_1$. According to (128) and (151), we now have the stress

$$\sigma^{\alpha\beta} = \mu\,A^{\alpha\beta} - q\,a^{\alpha\beta} - 4c\,H\,b^{\alpha\beta} \,, \tag{210}$$

due to stretching and bending. According to (152), the bending moment for the area-constrained case is the same as in Eqs. (198) and (199). From (73) then follows that the nonzero components of $N^\alpha_\beta = N^{\alpha\gamma}a_{\gamma\beta}$ are

$$\begin{aligned} N^1_1 &= \mu\,\lambda_1^2 - q - c\,\kappa_1^2 \,, \\ N^2_2 &= \mu\,\lambda_2^2 - q \,. \end{aligned} \tag{211}$$

As before, we examine the following two special cases:

1. Dirichlet case: Prescribing the deformation such that $\lambda_1 = \lambda_2 = 1$, we have

$$\begin{aligned} N^1_1 &= \mu - q - c\,\kappa^2 \,, \\ N^2_2 &= \mu - q \,. \end{aligned} \tag{212}$$

Due to the area constraint only two opposing edges need to be fixed, while the other two opposing edges can be kept free. Therefore we can either have $N^1_1 = 0$, for which $q = \mu - c\,\kappa_1^2$ and $N^2_2 = c\,\kappa_1^2$, or we can have $N^2_2 = 0$, for which $q = \mu$ and $N^1_1 = -c\,\kappa_1^2$. If all edges are fixed,

q remains unspecified. In order to equilibrate the structure, the surface pressure $p = N_1^1/R$ is required.

2. Neumann case: Keeping all edges free yields $N_1^1 = N_2^2 = 0$, such that $q = \mu\,\lambda_2^2 = \mu\,\lambda_1^{-2}$ according to (211b). Eq. (211a) then gives

$$\lambda_1^4 - \frac{c}{\mu}\kappa_1^2\lambda_1^2 - 1 = 0\ , \tag{213}$$

which has the positive real solution $\lambda_1 = \sqrt{a_0}$, where a_0 is from Eq. (208) above.

7.3 Unconstrained liquid shell

Let us now consider the unconstrained liquid shell model according to Secs. 5.2.1, 5.2.2 and 5.3.1. According to (130), (134), (140) and (141), the stress and moment components now are

$$\begin{aligned} \sigma^{\alpha\beta} &= \big(k\,\Delta H^2 - \bar{k}\kappa + \gamma\big)\,a^{\alpha\beta} - 2k\,\Delta H\,b^{\alpha\beta}\ , \\ M^{\alpha\beta} &= \big(k\,\Delta H + 2\bar{k}H\big)\,a^{\alpha\beta} - \bar{k}\,b^{\alpha\beta}\ , \end{aligned} \tag{214}$$

where either γ is a given constant according to model (129) or $\gamma = K(J-1)$ according to model (133). In the present case, $\kappa = 0$ and $\Delta H = H = \kappa_1/2$, such that we find

$$N^\alpha_\beta = (kH^2 + \gamma)\,\delta^\alpha_\beta - kH\,b^\alpha_\beta\ , \tag{215}$$

which has the non-zero components

$$\begin{aligned} N_1^1 &= \gamma - \frac{k}{4}\kappa_1^2\ , \\ N_2^2 &= \gamma + \frac{k}{4}\kappa_1^2\ . \end{aligned} \tag{216}$$

From (201) we further find

$$M = k\,H\ . \tag{217}$$

For given κ_1, the following four cases can now be identified:

1. All sides are fixed, i.e. λ_1 and λ_2 are given. N_1^1 and N_2^2 then follow from (216). In case of model (133), $\gamma = K(\lambda_1\lambda_2 - 1)$.
2. N_1^1 and λ_2 are given: γ and N_2^2 then follow from (216). Since γ cannot be an independent constant, only model (133) is in general permissible, yielding $\lambda_1 = (1 + \gamma/K)/\lambda_2$.
3. N_2^2 and λ_1 are given: Similar to case 2, γ and N_1^1 follow from (216) and $\lambda_2 = (1+\gamma/K)/\lambda_1$.
4. N_1^1 and N_2^2 are given. According to (216), this is generally not possible.

7.4 Area-constrained liquid shell

We finally consider the area-constrained liquid shell model according to Secs. 5.2.3 and 5.3.2. According to (137) and (149), the stress and moment components now are

$$\begin{aligned} \sigma^{\alpha\beta} &= -\big(2\bar{k}\kappa + q\big)\,a^{\alpha\beta} - 2k\,\Delta H\,b^{\alpha\beta}\ , \\ M^{\alpha\beta} &= \big(k\,\Delta H + 2\bar{k}H\big)\,a^{\alpha\beta} - \bar{k}\,b^{\alpha\beta}\ . \end{aligned} \tag{218}$$

With $\kappa = 0$ and $\Delta H = H = \kappa_1/2$ we find

$$N^\alpha_\beta = -q\,\delta^\alpha_\beta - kH\,b^\alpha_\beta\ , \tag{219}$$

which has the non-zero components

$$\begin{aligned} N^1_1 &= -q - \frac{k}{2}\kappa_1^2\ , \\ N^2_2 &= -q\ . \end{aligned} \tag{220}$$

From (201) we again find

$$M = k\,H\ . \tag{221}$$

Further, due to the free edges at $Y = 0$ and $Y = L$ we require that the bending moment M^{22} vanishes. This is the case for $\bar{k} = -k/2$.

For given κ_1, the following four BC cases can now be identified:

1. λ_1 and λ_2 are given (such that $\lambda_1\lambda_2 = 1$). N^1_1 and N^2_2 then follow from (220); q remains unspecified.
2. N^1_1 and λ_2 are given. q and N^2_2 then follow from (216), while $\lambda_1 = 1/\lambda_2$.
3. N^2_2 and λ_1 are given. Similar to case 2, q and N^1_1 follow from (216) while $\lambda_2 = 1/\lambda_1$.
4. N^1_1 and N^2_2 are given. According to (220), this is generally not possible.

8 Conclusion

This paper presents the governing equations of thin, quasi-static shells with either solid- or liquid-like constitutive behavior. Using Kirchhoff-Love assumptions, the shell kinematics during deformation is fully characterized by the change of the midplane tangent vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$, which is captured by the quantities $a_{\alpha\beta}$ and $b_{\alpha\beta}$. Starting from the balance laws for linear and angular momentum, the governing shell equations are then derived in strong form. Those are then complemented with general boundary conditions and general constitutive equations for hyperelastic materials. Various example models are provided for those, considering both solid and liquid shells and considering both unconstrained and area-constrained material behavior. For those models the material stability of the membrane and bending parts is assessed, and the weak form as well as its linearization is provided. As is shown, the weak form can be fully decomposed into in-plane and out-of-plane contributions. As an application of the theory, the paper finally presents the analytical solution for the homogenous bending and stretching of a flat sheet considering four different material models.

The theory and its application example presented here provide a rigorous basis for the development of numerical methods – for example in the framework of finite elements – that are applicable to both solid- and fluid-like material behavior. This is currently being considered in the framework of C^1-continuous surface discretizations (Duong et al., 2016; Sauer et al., 2016).

Acknowledgements

The authors are grateful to the German Research Foundation (DFG) for supporting this research under grants GSC 111 and SA1822/5-1.

References

Canham, P. B. (1970). The minimum energy of bending as a possible explanation of the biconcave shape of the human red blood cell. *J. Theoret. Biol.*, **26**:61–81.

Ciarlet, P. G. (2005). An introduction to differential geometry with applications to elasticity. *J. Elast.*, **78-79**:3–201.

Coleman, B. D. and Noll, W. (1964). The thermodynamics of elastic materials with heat conduction and viscosity. *Arch. Ration. Mech. Anal.*, **13**:167–178.

Duong, T. X., Roohbakhshan, F., and Sauer, R. A. (2016). A new rotation-free isogeometric thin shell formulation and a corresponding continuity constraint for patch boundaries. *Comput. Methods Appl. Mech. Engrg.*, published online, DOI: 10.1016/j.cma.2016.04.008.

Helfrich, W. (1973). Elastic properties of lipid bilayers: Theory and possible experiments. *Z. Naturforsch.*, **28c**:693–703.

Jenkins, J. T. (1977). The equations of mechanical equilibrium of a model membrane. *SIAM J. Appl. Math.*, **32**(4):755–764.

Kiendl, J., Bletzinger, K.-U., Linhard, J., and Wüchner, R. (2009). Isogeometric shell analysis with Kirchhoff-Love elements. *Comput. Methods Appl. Mech. Engrg.*, **198**:3902–3914.

Libai, A. and Simmonds, J. G. (1998). *The Nonlinear Theory of Elastic Shells.* Cambridge University Press, Cambridge, 2^{nd} edition.

Marsden, J. E. and Hughes, T. J. R. (1994). *Mathematical Foundations of Elasticity.* Dover, Mineola.

Naghdi, P. M. (1982). Finite deformation of elastic rods and shells. In Carlson, D. E. and Shields, R. T., editors, *Proceedings of the IUTAM Symposium on Finite Elasticity*, pages 47–103, The Hague. Martinus Nijhoff Publishers.

Pietraszkiewicz, W. (1989). Geometrically nonlinear theories of thin elastic shells. *Advances Mech.*, **12**(1):51–130.

Sauer, R. A. (2014). Stabilized finite element formulations for liquid membranes and their application to droplet contact. *Int. J. Numer. Meth. Fluids*, **75**(7):519–545.

Sauer, R. A. (2016). A contact theory for surface tension driven systems. *Math. Mech. Solids*, **21**(3):305–325.

Sauer, R. A., Duong, T. X., and Corbett, C. J. (2014). A computational formulation for constrained solid and liquid membranes considering isogeometric finite elements. *Comput. Methods Appl. Mech. Engrg.*, **271**:48–68.

Sauer, R. A., Duong, T. X., Mandadapu, K. K., and Steigmann, D. J. (2016). A stabilized finite element formulation for liquid shells and its application to lipid bilayers. *J. Comput. Phys.*, published online, DOI: 10.1016/j.jcp.2016.11.004.

Steigmann, D., Baesu, E., Rudd, R. E., Belak, J., and McElfresh, M. (2003). On the variational theory of cell-membrane equilibria. *Interfaces Free Bound.*, **5**:357–366.

Steigmann, D. J. (1999a). Fluid films with curvature elasticity. *Arch. Rat. Mech. Anal.*, **150**:127–152.

Steigmann, D. J. (1999b). On the relationship between the Cosserat and Kirchhoff-Love theories of elastic shells. *Math. Mech. Solids*, **4**:275–288.

Wriggers, P. (2006). *Computational Contact Mechanics.* Springer, 2$^{\text{nd}}$ edition.

Wriggers, P. (2008). *Nonlinear Finite Element Methods.* Springer.