The principle of virtual forces

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Principle of virtual forces:

If you apply infinitisemal small, virtual forces (stresses) on a field, the external virtual work is equal to the whole inner virtual work

The principle of virtual work is often used for calculation of displacements

The external work is defined as:

$$W_e = \int_{\Omega g} \mathbf{qu}\, dA + \int_B \mathbf{bu}\, dV$$

q – virtual surface forces
b – virtual volume forces
u – displacement condition

The internal work is defined as:

$$W_i = \int_B \boldsymbol{\sigma\varepsilon}\, dV$$

σ – virtual stresses
ϵ – strains

Consequentially:

$$W_i - W_e = \int_B \boldsymbol{\sigma\varepsilon}\, dV - \left[\int_{\Omega g} \mathbf{qu}\, dA + \int_B \mathbf{bu}\, dV\right] = 0$$

That means for a simple calculation of a beam structure:

Internal Work:

$$
\begin{aligned}
W_i = &\int_0^l \overline{N}\left[\frac{N}{EA}(1+\varphi) + \alpha T + \varepsilon\right]dx + \int_0^l \overline{Q}\left[\frac{Q}{GA}(1+\varphi)\right]dx \\
&+ \int_0^l \overline{M}\left[\frac{M}{EI}(1+\varphi) + \alpha\frac{\Delta T}{h} + \frac{\Delta\varepsilon}{h}\right]dx \\
&+ \sum\nolimits_A \overline{R}\,\frac{R}{c_N} + \sum\nolimits_A \overline{M}\,\frac{M}{c_M} + \sum\nolimits_S \overline{Q}\,\Delta u + \sum\nolimits_S \overline{M}\,\Delta\varphi
\end{aligned}
$$

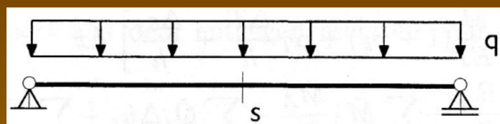
External Work:

$$W_e = W_e^w + W_e^x$$

$$W_e^w = \sum \bar{A}_H u_A + \sum \bar{A}_V w_A + \sum \bar{M}_A \varphi_A$$

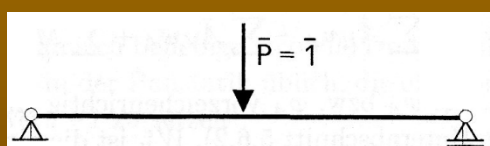
$$W_e^X = \begin{cases} \bar{1} u_s \text{ for displacement } u_s \\ \bar{1} w_s \text{ for displacement } w_s \\ \bar{1} \varphi_s \text{ for rotation } \varphi_s \\ \bar{1} \Delta w_s \text{ for relativ displacement } \Delta w_s \\ \bar{1} \Delta \varphi_s \text{for relativ rotation } \Delta \varphi_s \end{cases}$$

A simple example:

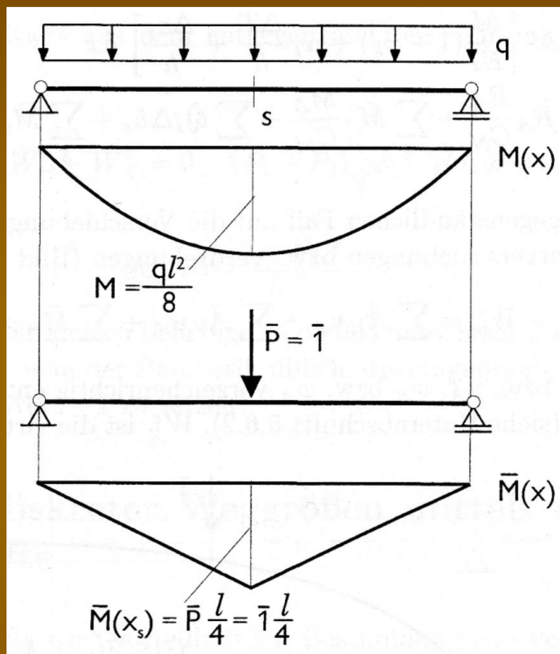


Statically defined, shear fixed beam with
a constant distributed load q
That means: N = Q = 0

We are looking for the displacement w at the point s



To get this displacement,
we bring up a unity force at point s in the
direction of the displacement we are looking for



Now we have to calculate seperately the moment diagrams for both loads and superpose it

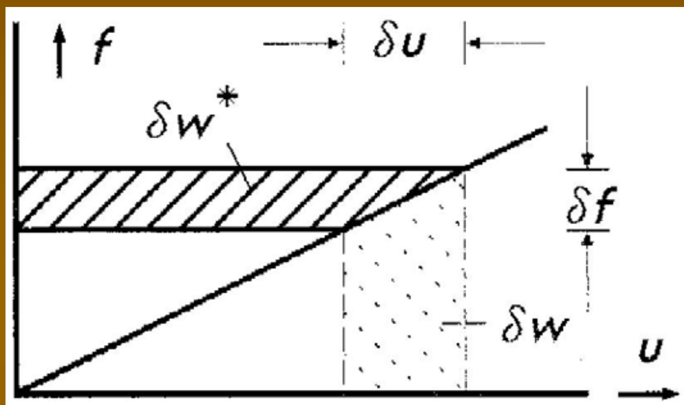
The result is the searched displacement w

Differences between the principle of virtual forces and the priciple of virtual displacement

Principle of virtual forces	Principle of virtual displacement
• Simple manual calculation for easy systems • equilibrium conditions achieved exactly • leads to larger displacements than real => soft solution	• easier to program for complex systems • kinematic conditions achieved exactly • leads to smaller displacements than real => stiff solution

Theoretical involvement in FEM

Complementary work δw*:



Force-displacement diagramm

δu = virtual displacement

δw = virtual work

δf = virtual force

Definition of complementary work:

$$\delta w^{*} = \sum_{a} u_{a} \, \delta f_{a} \quad ; \quad (a = 1 \ldots n) \quad ,$$

$$\delta \pi^{*} = \sum_{i,j} \int_{V} \varepsilon_{ij} \, \delta \sigma_{ij} \, \mathrm{d}v \quad ; \quad (i,j = 1 \ldots c) \; .$$

For a linear problem, like we assumed before, the complementary works are equal to the real works.

That means.:

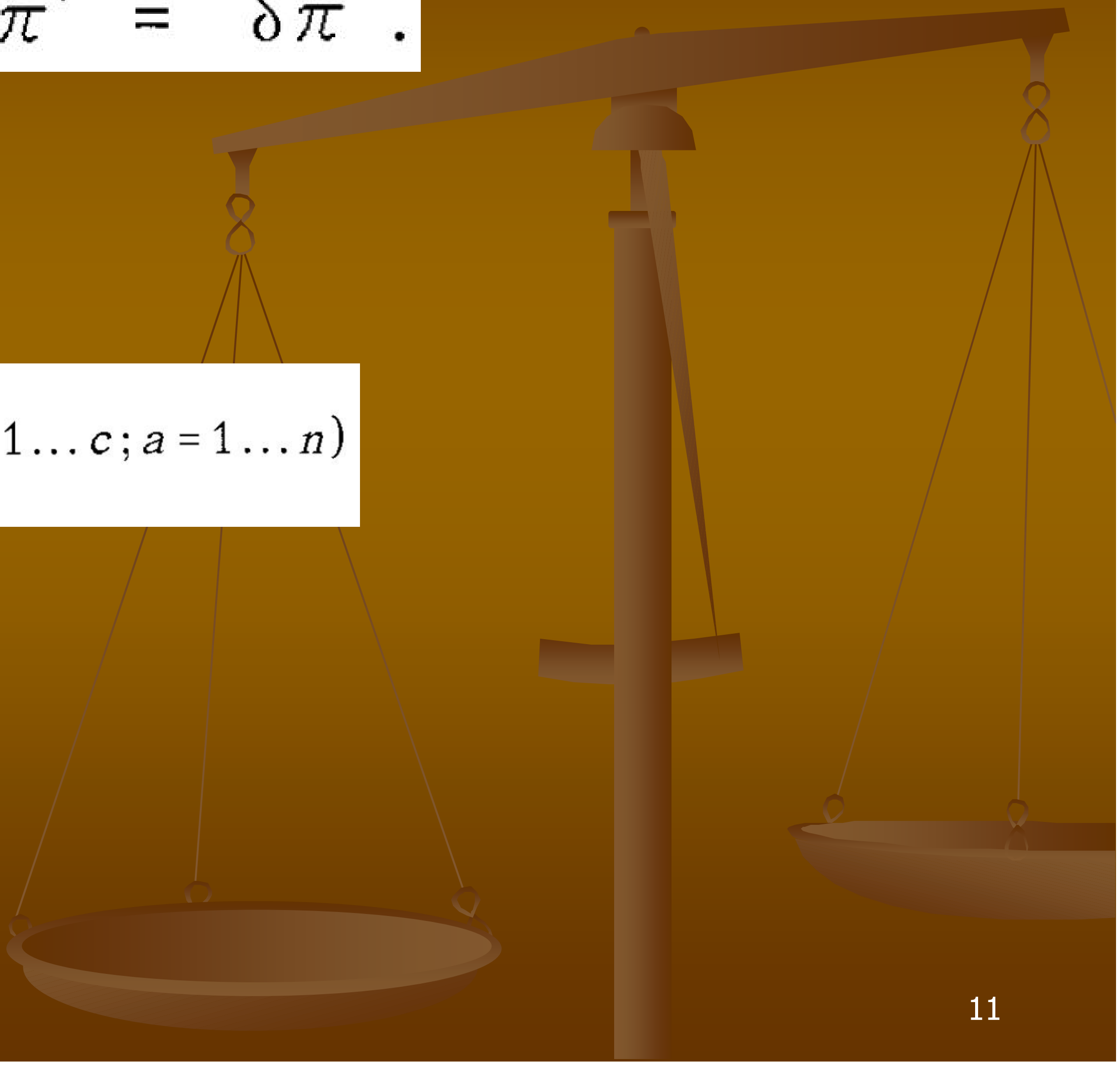
$$\delta w^{*} = \delta w \;; \qquad \delta \pi^{*} = \delta \pi \;.$$

It is essential:

$$\delta \sigma_{ij} = \sum_{a} \frac{\partial \sigma_{ij}}{\partial f_{a}} \, \delta f_{a} \qquad ; \quad (i,j = 1 \ldots c \,;\, a = 1 \ldots n)$$

Equilibrium of Work:

$$\delta w^{*} = \delta \pi^{*}$$



From this it follows that:

$$u_a = \sum_{i,j} \int_V \frac{\partial \sigma_{ij}}{\partial f_a} \varepsilon_{ij} \, \mathrm{d}v \quad ; \quad (i,j = 1 \ldots c ; a = 1 \ldots n)$$

Force - displacement law:

$$u_a = \sum_b c_{ab} f_b \quad ; \quad (a, b = 1 \ldots n)$$

Both equations leads us to the following flexibility matrix:

$$c_{ab} = \frac{\partial u_a}{\partial f_b} = \frac{\partial}{\partial f_b} \left(\sum_{i,j} \int_V \frac{\partial \sigma_{ij}}{\partial f_a} \varepsilon_{ij} \, \mathrm{d}v \right)$$

In Consideration of a linear problem:

$$c_{ab} = \sum_{i,j} \int_{V} \frac{\partial \sigma_{ij}}{\partial f_a} \frac{\partial \varepsilon_{ij}}{\partial f_b} \, \mathrm{d}v \quad ; \quad (i,j = 1 \ldots c ; a,b = 1 \ldots n)$$

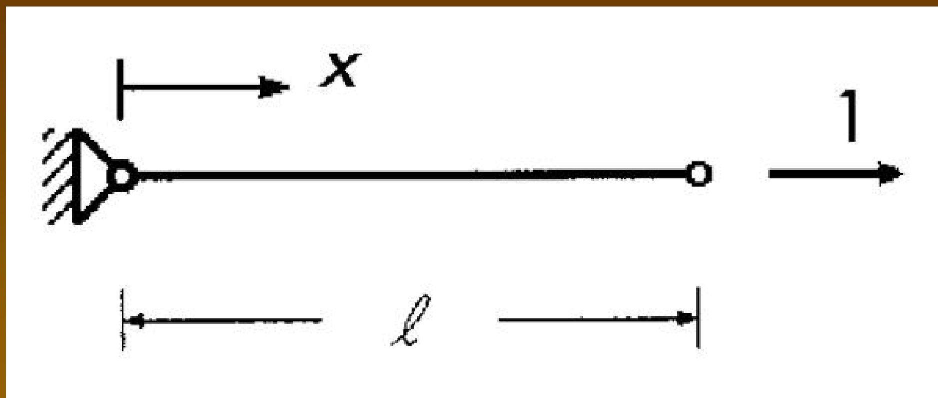
By using Hooke´s law:

$$c_{ab} = \sum_{i,j,k,l} \int_{V} \frac{\partial \sigma_{ij}}{\partial f_a} c_{ijkl} \frac{\partial \sigma_{kl}}{\partial f_b} \, \mathrm{d}v \; ; \quad (i,j,k,l = 1 \ldots c ; a,b = 1 \ldots n)$$

Like the stiffness matrix, the flexibility matrix is symmetrically:

$$c_{ab} = c_{ba} \quad ; \quad (a,b = 1 \ldots n)$$

A little example:



A system with one degree of freedom which leads to a flexibility matrix with one coefficient:

$$c_{11} = A \int_{\ell} \frac{\partial \sigma_{11}}{\partial f_1} c_{1111} \frac{\partial \sigma_{11}}{\partial f_1} \, \mathrm{d}x$$

Because of $c_{1111} = k_{1111} = E$:

$$c_{11} = \frac{A}{E} \int_{\ell} \left(\frac{\partial \sigma}{\partial f} \right)^2 dx \qquad ; \quad (a, b = 1, 2)$$

The stress results of the load f in the first degree of freedom:

$$\sigma = \frac{f}{A}$$

So we get the flexibility coefficient we are looking for:

$$c_{11} = \frac{1}{AE} \int_{\ell} dx = \frac{\ell}{AE}$$

Quintessence

The problem of using the principle of virtual forces in finite element programs is, that you can only calculate stresses from external forces, if you have a statically defined system.

In all other cases you have to manipulate the system to get a primary structure (statically defindded system) by reducing the degree of freedom.

This manipulation is a big problem for an automatical process and its much easier to program it with the principle of virtual displacement.

Nevertheless, it is possible using the principle of virtual forces, as you can see on the example before.

Thanks for your attention!