Shape Function Generation and Requirements

Requirements

(A) Interpolation

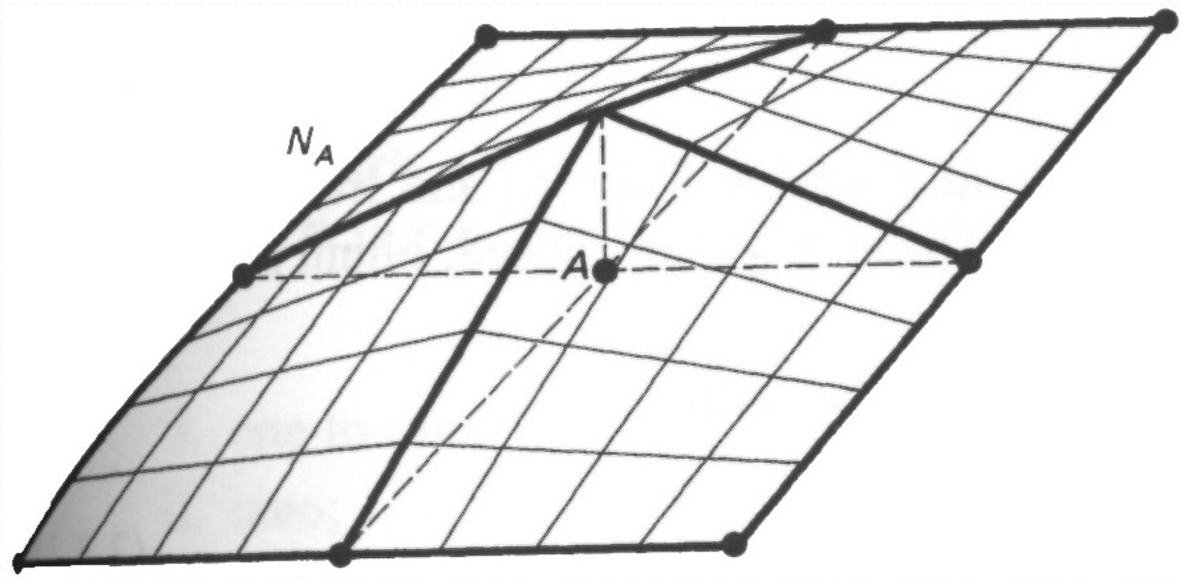
(B) Local Support

(C) Continuity (Intra- & Inter-Element)

(D) Completeness

Requirements

- (A) *Interpolation condition*. Takes a unit value at node *i* , and is zero at all other nodes.



Requirements

- (B) *Local support condition*. Vanishes over any element boundary (a side in 2D, a face in 3D) that does not include node *i* .
- (C) *Interelement compatibility condition*. Satisfies *C*0 continuity between adjacent elements over any element boundary that includes node *i* .
- (D) *Completeness condition*. The interpolation is able to represent exactly any displacement field which is a linear polynomial in *x* and *y*; in particular, a constant value.
- **If (C) and (D) are considered together,this case can be called CONSISTENCY.**

The Variational Index m

Bar

$$\Pi[u] = \int_0^L \left(\tfrac{1}{2}\, u' E A u' - q u\right)\, dx$$

$m = 1$

Beam

$$\Pi[v] = \int_0^L \left(\tfrac{1}{2}\, v'' E I v'' - q v\right)\, dx$$

$m = 2$

Requirements

- What are the minimum requirements that the finite element shape functions must show so that convergence is assured.

Two have been accepted for a long time:

Completeness

The ***element shape functions*** must represent exactly all polynomial terms of order $\leq m$ in the Cartesian coordinates. A set of shape functions that satisfies this condition is call m-complete

Compatibility

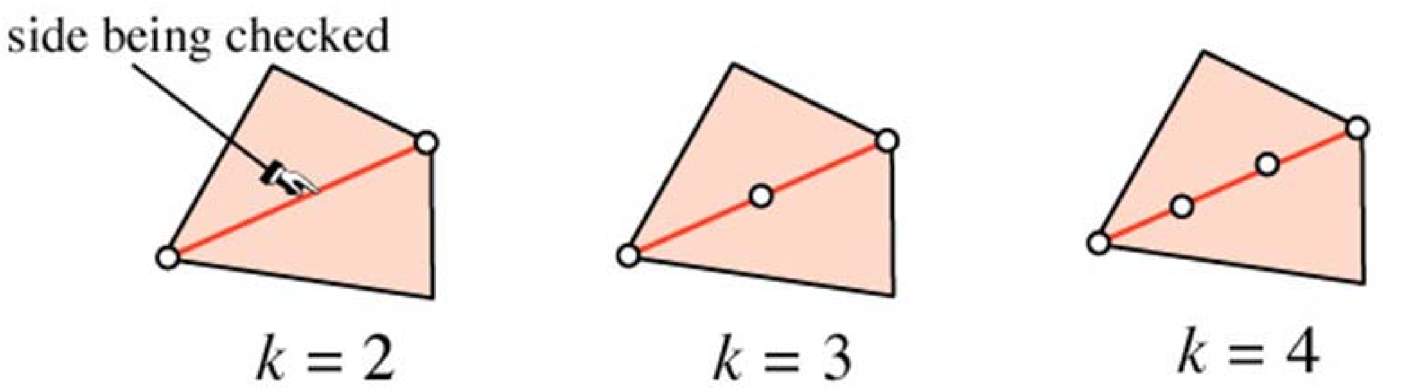
The ***patch trial functions*** must be $C^{(m-1)}$ continuous between elements, and C^{m} piecewise differentiable inside each element

Completeness & Compatibility in Terms of m

Continuity (which is the toughest to meet!)

- A structure is sub-divided into sub-regions or elements. The overall deformation of the structure is built-up from the values of the displacements at the nodes that form the net or grid and the shape functions within elements.

Let k be the number of nodes on a side:

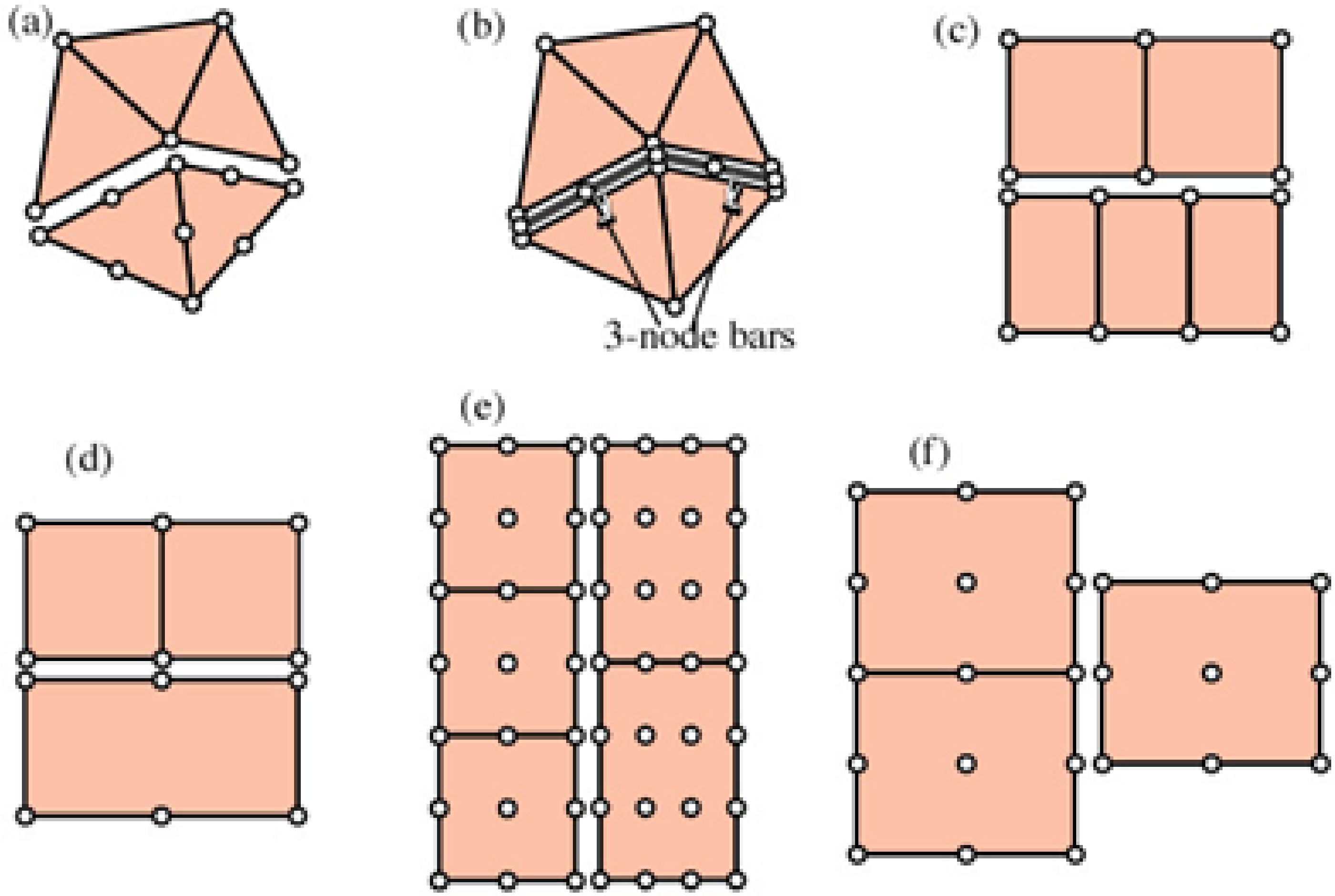


The variation of each element shape function along the side must be of polynomial order $k - 1$

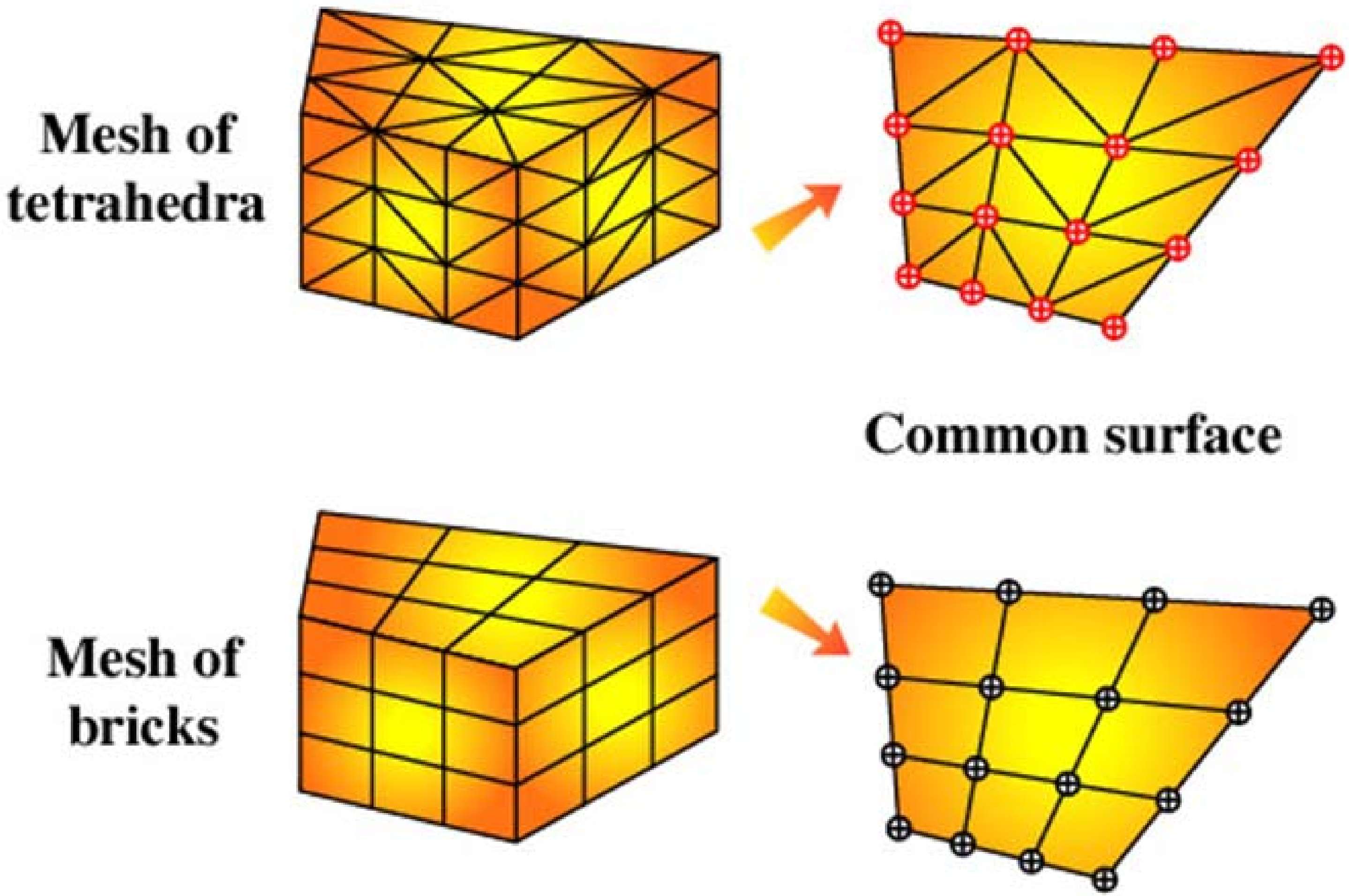
If ***more, continuity is violated***

If ***less, nodal configuration is wrong*** (too many nodes)

2D Nonmatching Mesh Examples



3D Nonmatching Mesh Example



Completeness

In the finite element method, or for that matter,in any approximate method, we are trying to replace an unknown function $\varnothing(x)$, which is the exact solution to a boundary value problem over a domain enclosed by a boundary by an approximate function $\varnothing(x)$ which is constituted from a set of shape or basis functions.

Generation of Shape Functions

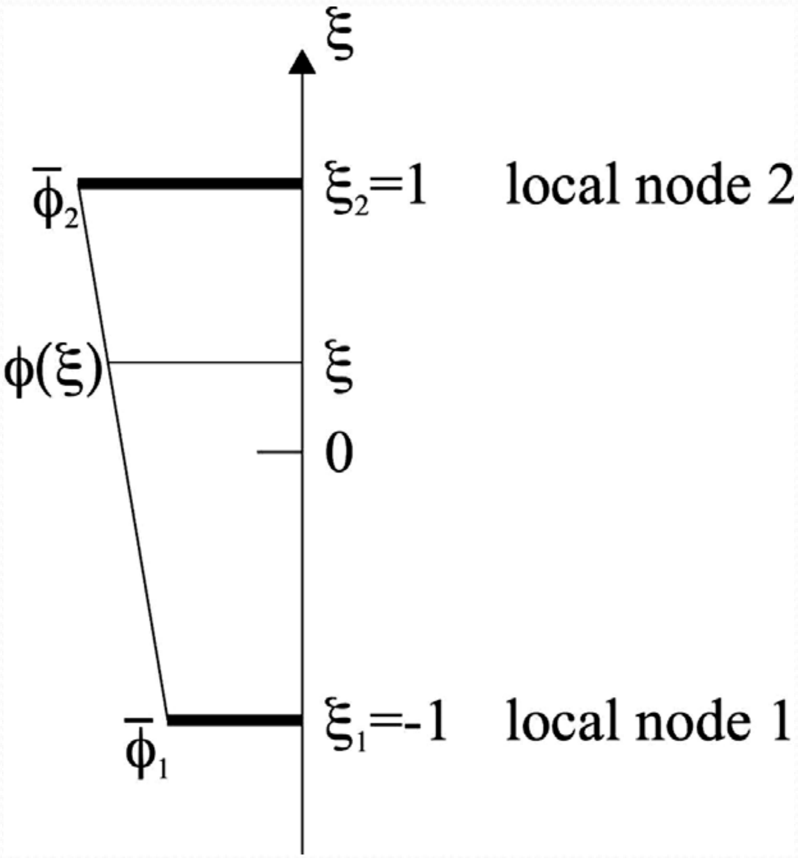
- Generation of shape functions is the most fundamental task in any finite element implementation.
- How isoparametric shape functions can be directly constructed by geometric considerations;

- Traditional interpolation takes the following steps

- 1. Choose a interpolation function
- 2. Evaluate interpolation function at known points
- 3. Solve equations to determine unknown constants

- Ø=[X] {a}   Øe=[A] {a}

Local Interpolation (1D)



$$\phi(\xi) = \alpha_1 + \alpha_2 \xi$$

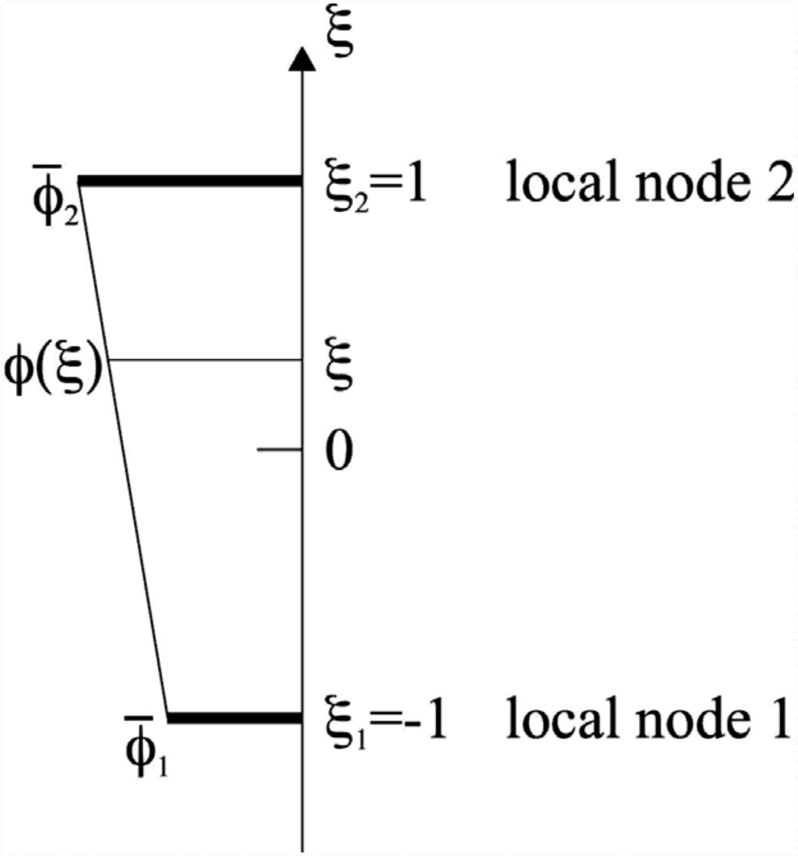
$$\begin{pmatrix} \bar{\phi}_1 \\ \bar{\phi}_2 \end{pmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{pmatrix} \bar{\phi}_1 \\ \bar{\phi}_2 \end{pmatrix}$$

$$\phi(\xi) = \frac{\bar{\phi}_1 + \bar{\phi}_2}{2} + \frac{-\bar{\phi}_1 + \bar{\phi}_2}{2} \xi$$

$$\phi(\xi) = \frac{1-\xi}{2} \bar{\phi}_1 + \frac{1+\xi}{2} \bar{\phi}_2$$

Local Shape Function (1D)



$$\phi(\xi) = \frac{1-\xi}{2}\bar{\phi}_1 + \frac{1+\xi}{2}\bar{\phi}_2$$

$$\phi(\xi) = N^{(1)}(\xi)\bar{\phi}_1 + N^{(2)}(\xi)\bar{\phi}_2$$

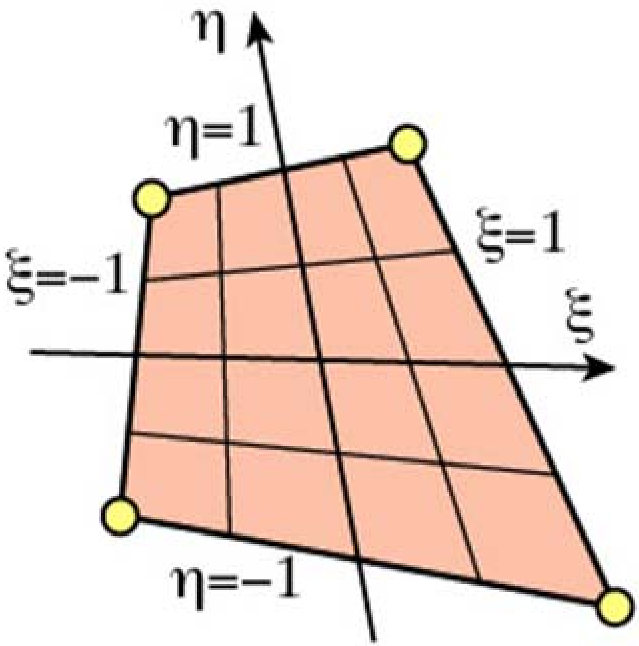
$$N^{(1)}(\xi) = \frac{1-\xi}{2}, \quad N^{(2)}(\xi) = \frac{1+\xi}{2}$$

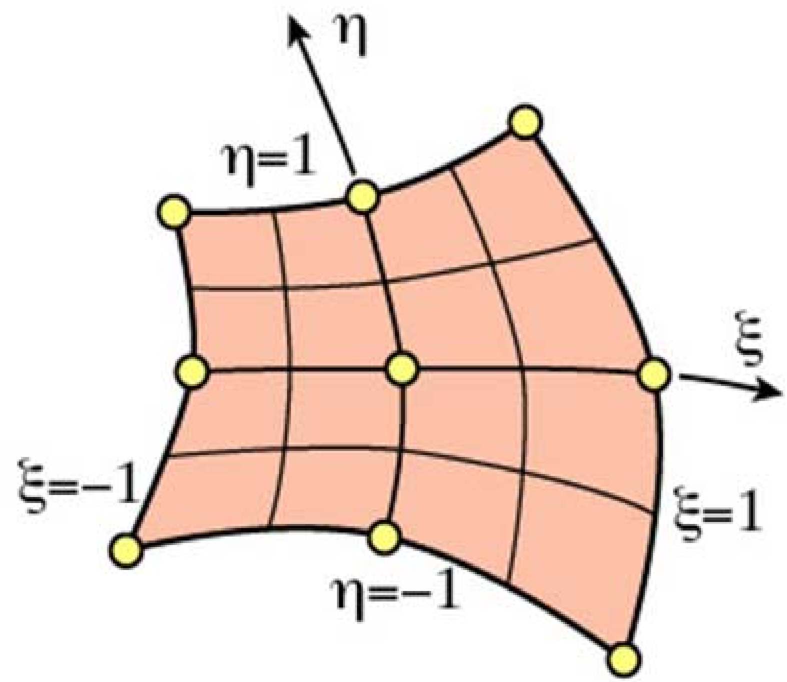
$$\bar{\xi}^{(1)} = -1$$

$$\bar{\xi}^{(2)} = 1$$

$$N^{(n)}(\xi) = \frac{1+\bar{\xi}^{(n)}\xi}{2}, \quad n = 1,2$$

Quadrilateral Coordinates ξ, η



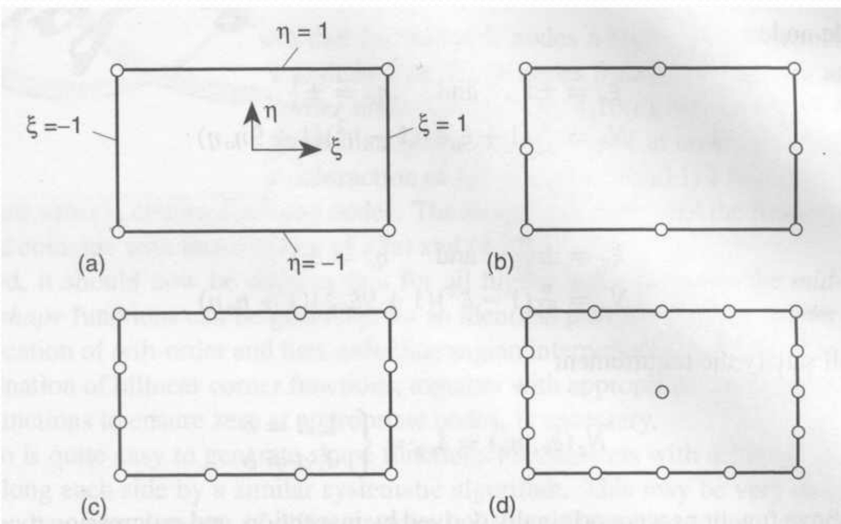


<u>4 Node Bilinear Quadrilateral</u> **<u>9 Node Biquadratic Quadrilateral</u>**

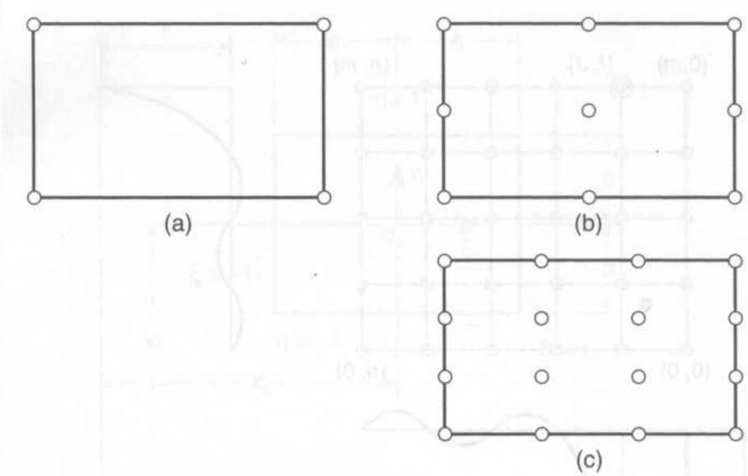
Quadrilateral Elements

Higher Order Rectangular Elements

- More nodes; still 2 translational d.o.f. per node.
- “Higher order” ⇒ higher degree of complete polynomial contained in displacement approximations.
- Two general “families” of such elements:



Serendipity

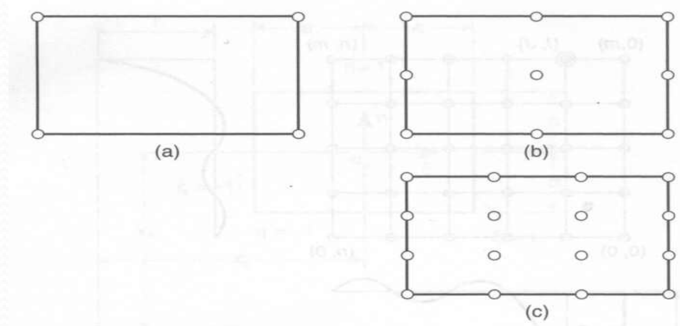


Lagrangian

Quadrilateral Elements

Lagrangian Elements:

- Order n element has $(n+1)^2$ nodes arranged in square-symmetric pattern – requires internal nodes.

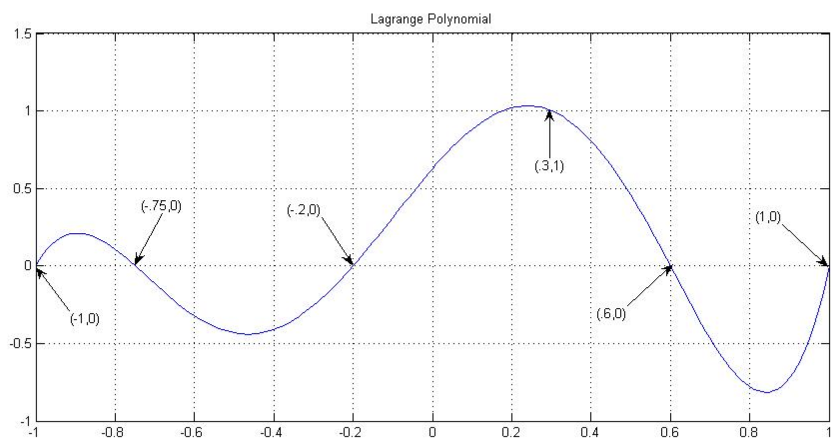


- Shape functions are products of nth order polynomials in each direction. ("biquadratic", "bicubic", ...)
- Bilinear quad is a Lagrangian element of order $n = 1$.

Quadrilateral Elements

Lagrangian Shape Functions:

- Uses a procedure that automatically satisfies the Kronecker delta property for shape functions.
  - Consider 1D example of 6 points; want function = 1 at $\xi_3 = 0.3$ and function = 0 at other designated points:



$$\xi_0 = -1;$$
$$\xi_1 = -.75;$$
$$\xi_2 = -.2;$$
$$\xi_3 = .3;$$
$$\xi_4 = .6;$$
$$\xi_5 = 1.$$

$$L_3^{(5)}(\xi) = \frac{(\xi - \xi_0)(\xi - \xi_1)(\xi - \xi_2)(\xi - \xi_4)(\xi - \xi_5)}{(\xi_3 - \xi_0)(\xi_3 - \xi_1)(\xi_3 - \xi_2)(\xi_3 - \xi_4)(\xi_3 - \xi_5)}.$$

Quadrilateral Elements

Lagrangian Shape Functions:

- Can perform this for any number of points at any designated locations.

$$L_k^{(m)}(\xi) = \frac{(\xi-\xi_0)(\xi-\xi_1)\ldots(\xi-\xi_{k-1})(\xi-\xi_{k+1})\ldots(\xi-\xi_m)}{(\xi_k-\xi_0)(\xi_k-\xi_1)\ldots(\xi_k-\xi_{k-1})(\xi_k-\xi_{k+1})\ldots(\xi_k-\xi_m)} = \prod_{\substack{i=0 \\ i\neq k}}^{m} \frac{(\xi-\xi_i)}{(\xi_k-\xi_i)}.$$

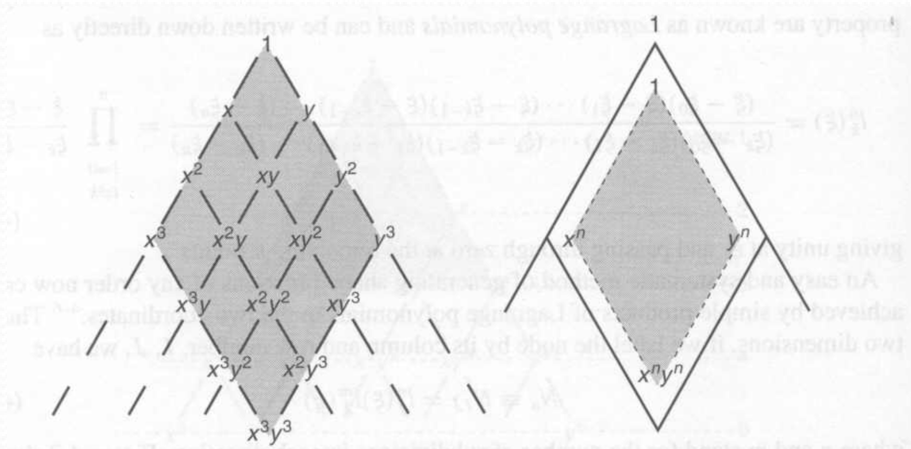
No ξ-ξ_k term!

Lagrange polynomial of order *m* at node *k*

Quadrilateral Elements

Notes on Lagrangian Elements:

- Once shape functions have been identified, there are no procedural differences in the formulation of higher order quadrilateral elements and the bilinear quad.
- Pascal's triangle for the Lagrangian quadrilateral elements:

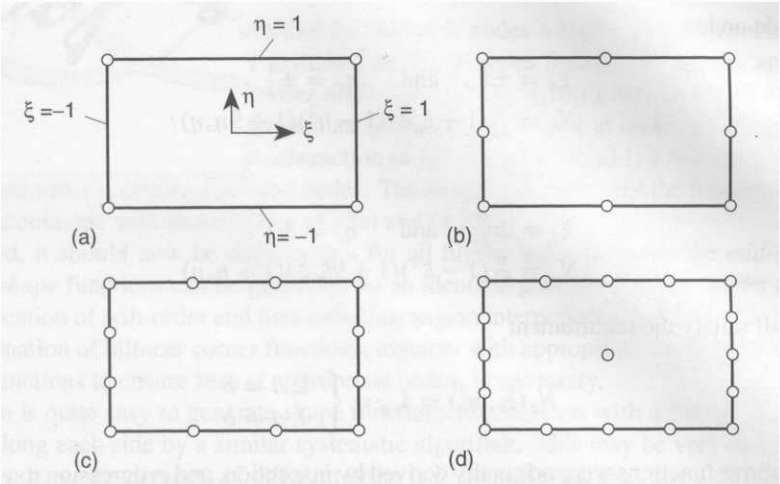


3 x 3 n x n

Quadrilateral Elements

Serendipity Elements:

- In general, only boundary nodes – avoids internal ones.

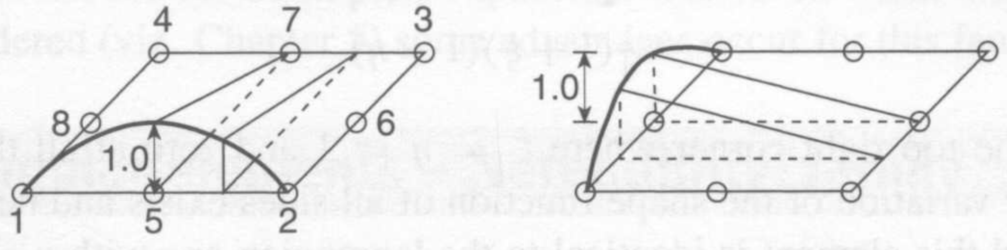


- Not as accurate as Lagrangian elements.
- However, more efficient than Lagrangian elements and avoids certain types of instabilities.

Quadrilateral Elements

Serendipity Shape Functions:

- Shape functions for **mid-side nodes** are products of an *n*th order polynomial parallel to side and a linear function perpendicular to the side.
  - E.g., quadratic serendipity element:

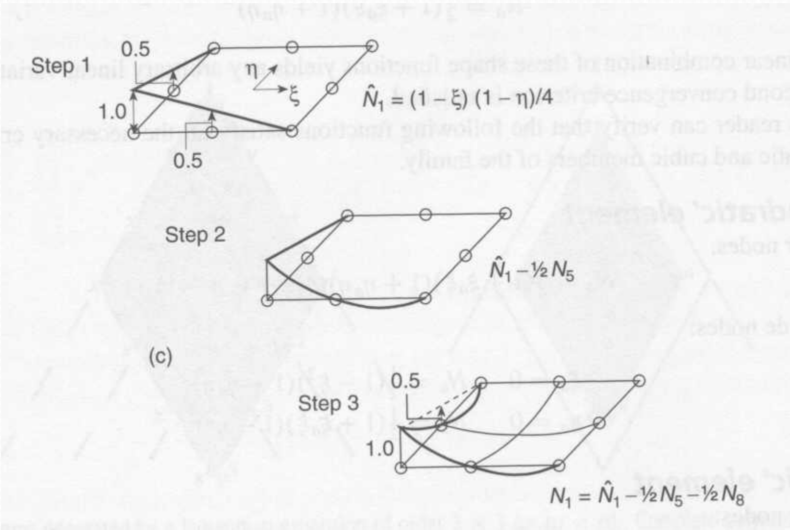


(a) $N_5 = \frac{1}{2}(1-\xi^2)(1-\eta)$ (b) $N_8 = \frac{1}{2}(1-\xi)(1-\eta^2)$

$$N_6 = \tfrac{1}{2}(1+\xi)(1-\eta^2);\, N_7 = \tfrac{1}{2}(1-\xi^2)(1+\eta).$$

Quadrilateral Elements

- Shape functions for **corner nodes** are modifications of the shape functions of the bilinear quad.
  - Step #1: start with appropriate bilinear quad shape function, $\hat{N}_1$
  - Step #2: subtract out mid-side shape function N_5 with appropriate weight $\hat{N}_1(\text{node } \#5) = \frac{1}{2}$
  - Step #3: repeat Step #2 using mid-side shape function N_8 and weight $\hat{N}_1(\text{node } \#8) = \frac{1}{2}$

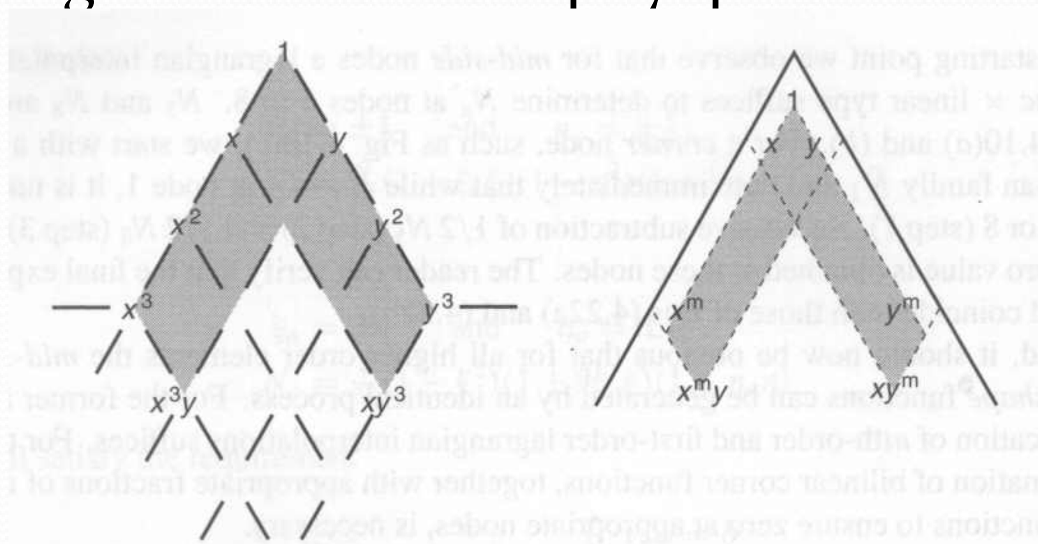


$$N_k = \tfrac{1}{4}(1+\xi_k\xi)(1+\eta_k\eta)(\xi_k\xi+\eta_k\eta-1); k = 1,2,3,4.$$

Quadrilateral Elements

Notes on Serendipity Elements:

- Once shape functions have been identified, there are no procedural differences in the formulation of higher order quadrilateral elements and the bilinear quad.
- Pascal's triangle for the serendipity quadrilateral elements:



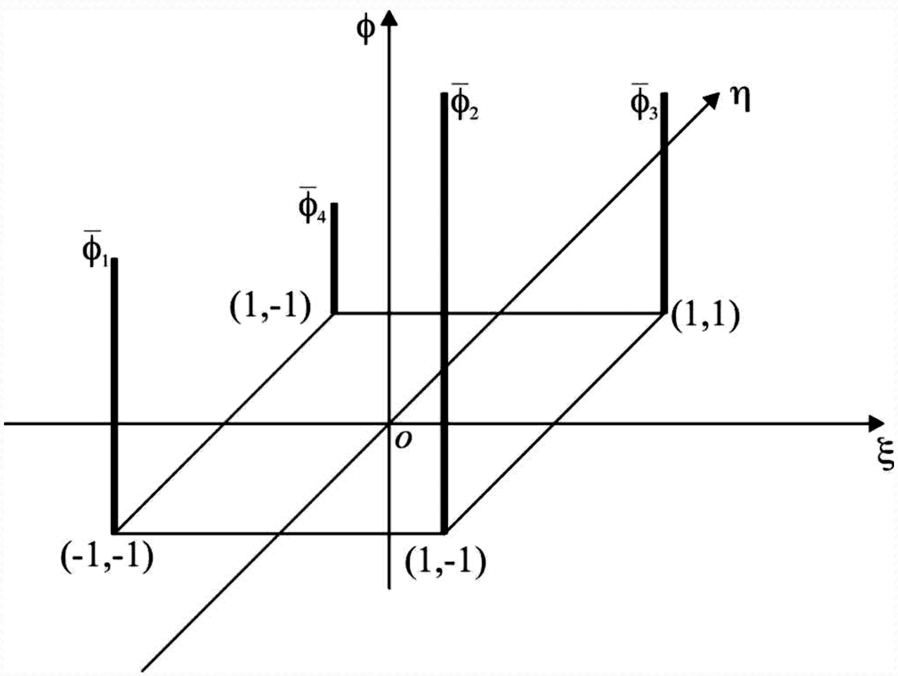
3 x 3 | m x m

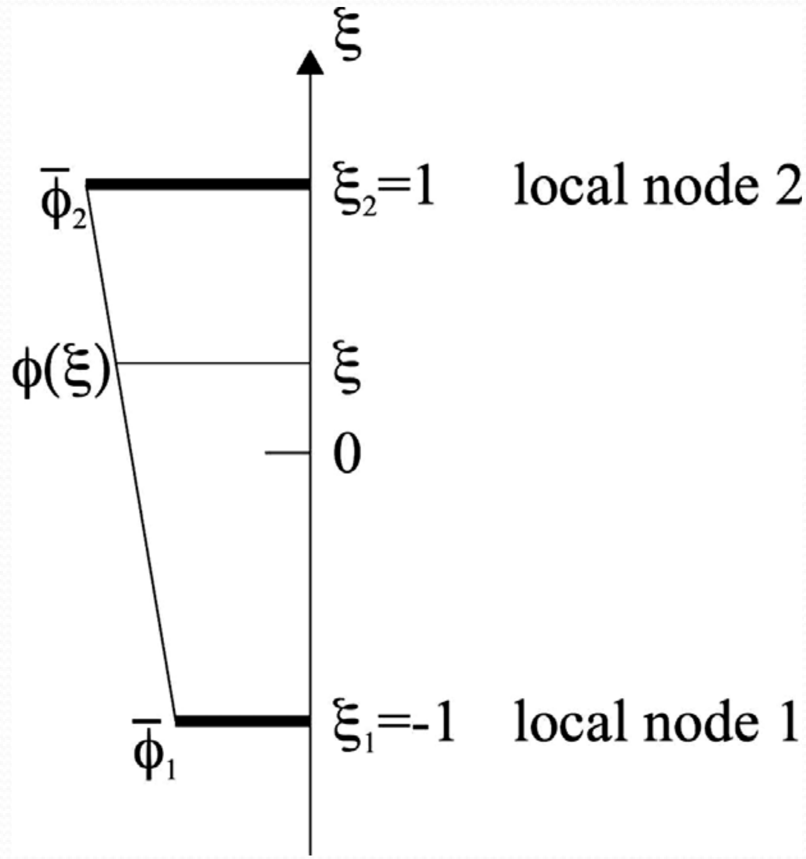
2D Shape Function

1D Shape Function

$$\phi(\xi,\eta) = \alpha_1 + \alpha_2\xi + \alpha_3\eta + \alpha_4\xi\eta$$

$$\phi(\xi) = \alpha_1 + \alpha_2\xi$$



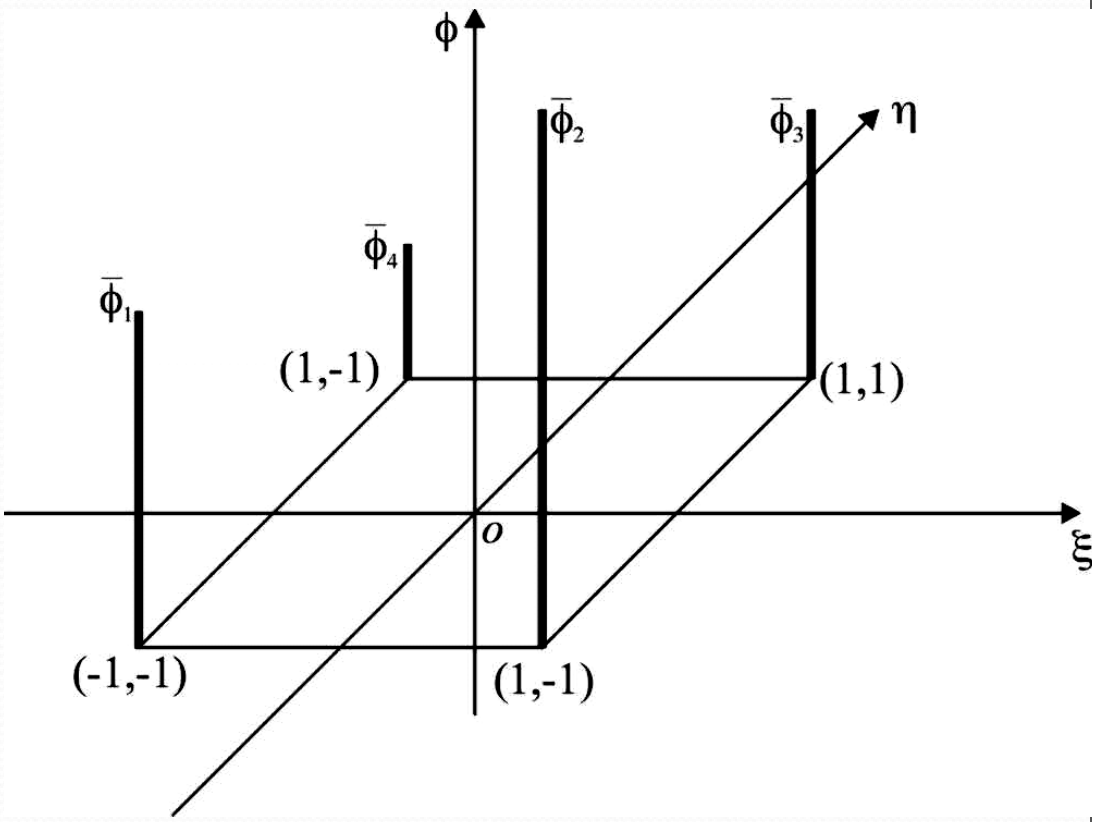


2D Interpolation Surface

$$\phi(\xi,\eta) = \alpha_1 + \alpha_2\xi + \alpha_3\eta + \alpha_4\xi\eta$$

$$\begin{pmatrix} \bar{\phi}_1 \\ \bar{\phi}_2 \\ \bar{\phi}_3 \\ \bar{\phi}_4 \end{pmatrix} = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix}$$

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \end{bmatrix} \begin{pmatrix} \bar{\phi}_1 \\ \bar{\phi}_2 \\ \bar{\phi}_3 \\ \bar{\phi}_4 \end{pmatrix}$$



Interpolation Surface

$$\phi(\xi,\eta)=\alpha_1+\alpha_2\xi+\alpha_3\eta+\alpha_4\xi\eta$$

$$\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{pmatrix}=\frac{1}{4}\begin{bmatrix}1 & 1 & 1 & 1\\ -1 & 1 & 1 & -1\\ -1 & -1 & 1 & 1\\ 1 & -1 & 1 & -1\end{bmatrix}\begin{pmatrix}\bar{\phi}_1\\ \bar{\phi}_2\\ \bar{\phi}_3\\ \bar{\phi}_4\end{pmatrix}$$

$$\phi(\xi,\eta)=\frac{\bar{\phi}_1+\bar{\phi}_2+\bar{\phi}_3+\bar{\phi}_4}{4}+\frac{-\bar{\phi}_1+\bar{\phi}_2+\bar{\phi}_3-\bar{\phi}_4}{4}\xi+\frac{-\bar{\phi}_1-\bar{\phi}_2+\bar{\phi}_3+\bar{\phi}_4}{4}\eta+\frac{\bar{\phi}_1-\bar{\phi}_2+\bar{\phi}_3-\bar{\phi}_4}{4}\xi\eta$$

$$\phi(\xi,\eta)=\left(\frac{1-\xi-\eta+\xi\eta}{4}\right)\bar{\phi}_1+\left(\frac{1+\xi-\eta-\xi\eta}{4}\right)\bar{\phi}_2+\left(\frac{1+\xi+\eta+\xi\eta}{4}\right)\bar{\phi}_3+\left(\frac{1-\xi+\eta-\xi\eta}{4}\right)\bar{\phi}_4$$

Shape Function (2D)

$$\phi(\xi,\eta)=\left(\frac{1-\xi-\eta+\xi\eta}{4}\right)\overline{\phi}_1+\left(\frac{1+\xi-\eta-\xi\eta}{4}\right)\overline{\phi}_2+\left(\frac{1+\xi+\eta+\xi\eta}{4}\right)\overline{\phi}_3+\left(\frac{1-\xi+\eta-\xi\eta}{4}\right)\overline{\phi}_4$$

$$\phi(\xi,\eta)=N^{(1)}\overline{\phi}_1+N^{(2)}\overline{\phi}_2+N^{(3)}\overline{\phi}_3+N^{(4)}\overline{\phi}_4$$

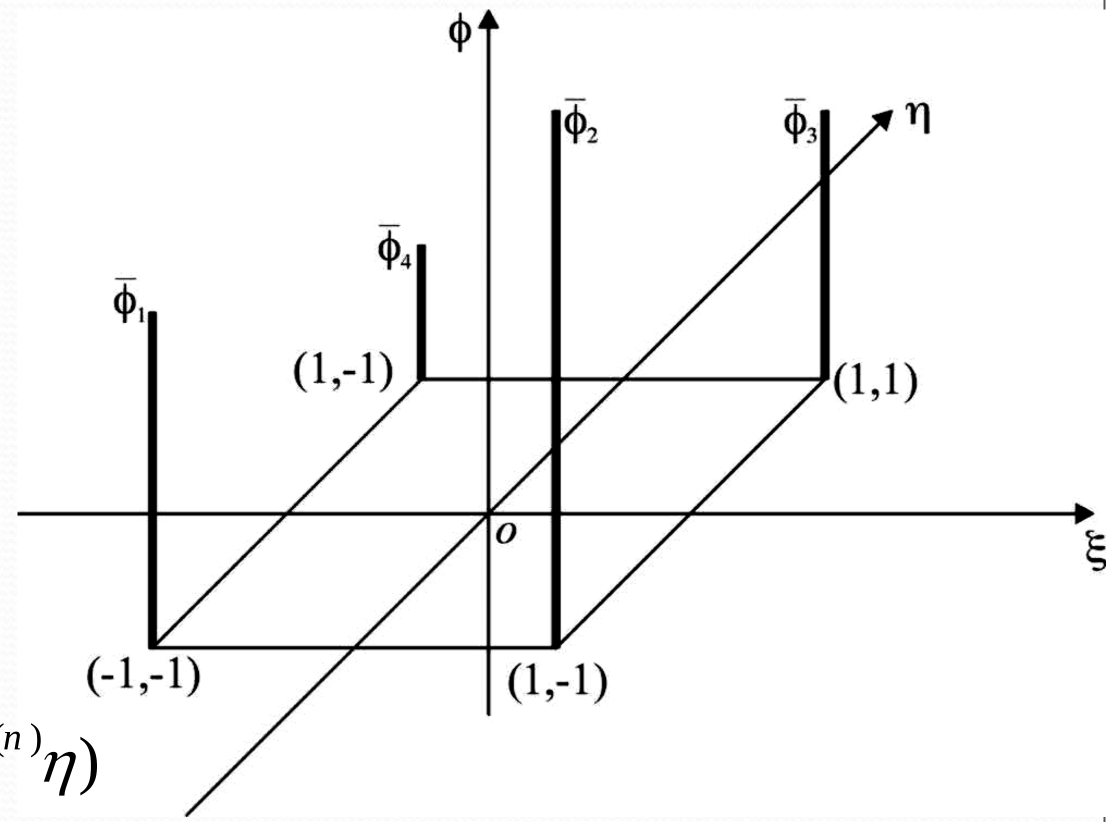
$$N^{(1)}=\frac{1}{4}(1-\xi)(1-\eta)$$

$$N^{(2)}=\frac{1}{4}(1+\xi)(1-\eta)$$

$$N^{(3)}=\frac{1}{4}(1+\xi)(1+\eta)$$

$$N^{(4)}=\frac{1}{4}(1-\xi)(1+\eta)$$

$$N^{(n)}(\xi,\eta)=\frac{1}{4}(1+\overline{\xi}^{(n)}\xi)(1+\overline{\eta}^{(n)}\eta)$$



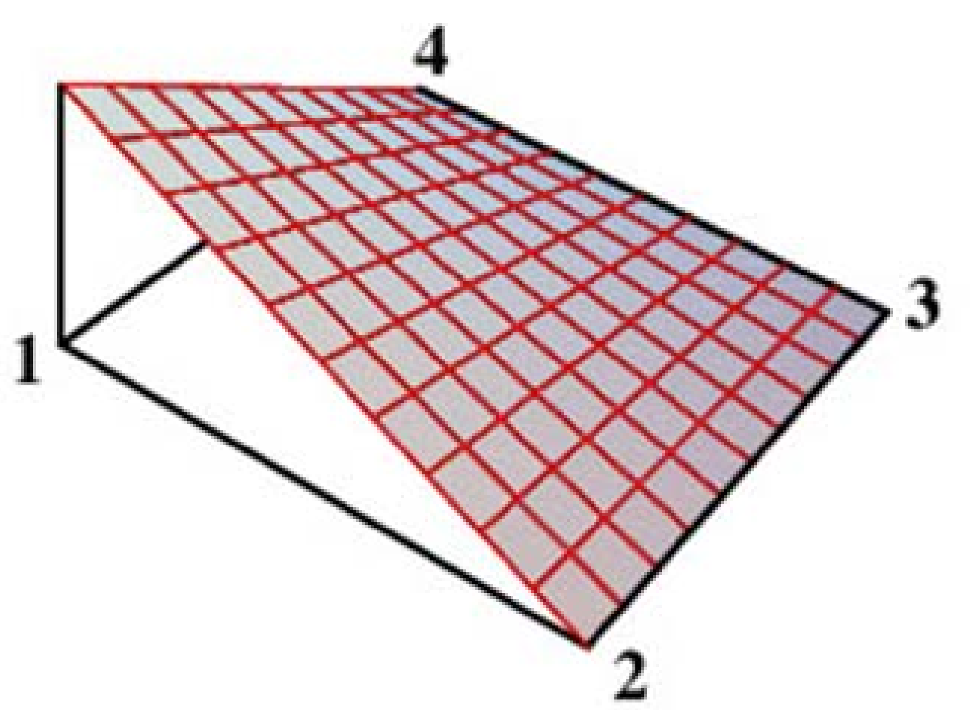
Bilinear Shape Function

$$N^{(1)} = \frac{1}{4}(1-\xi)(1-\eta)$$

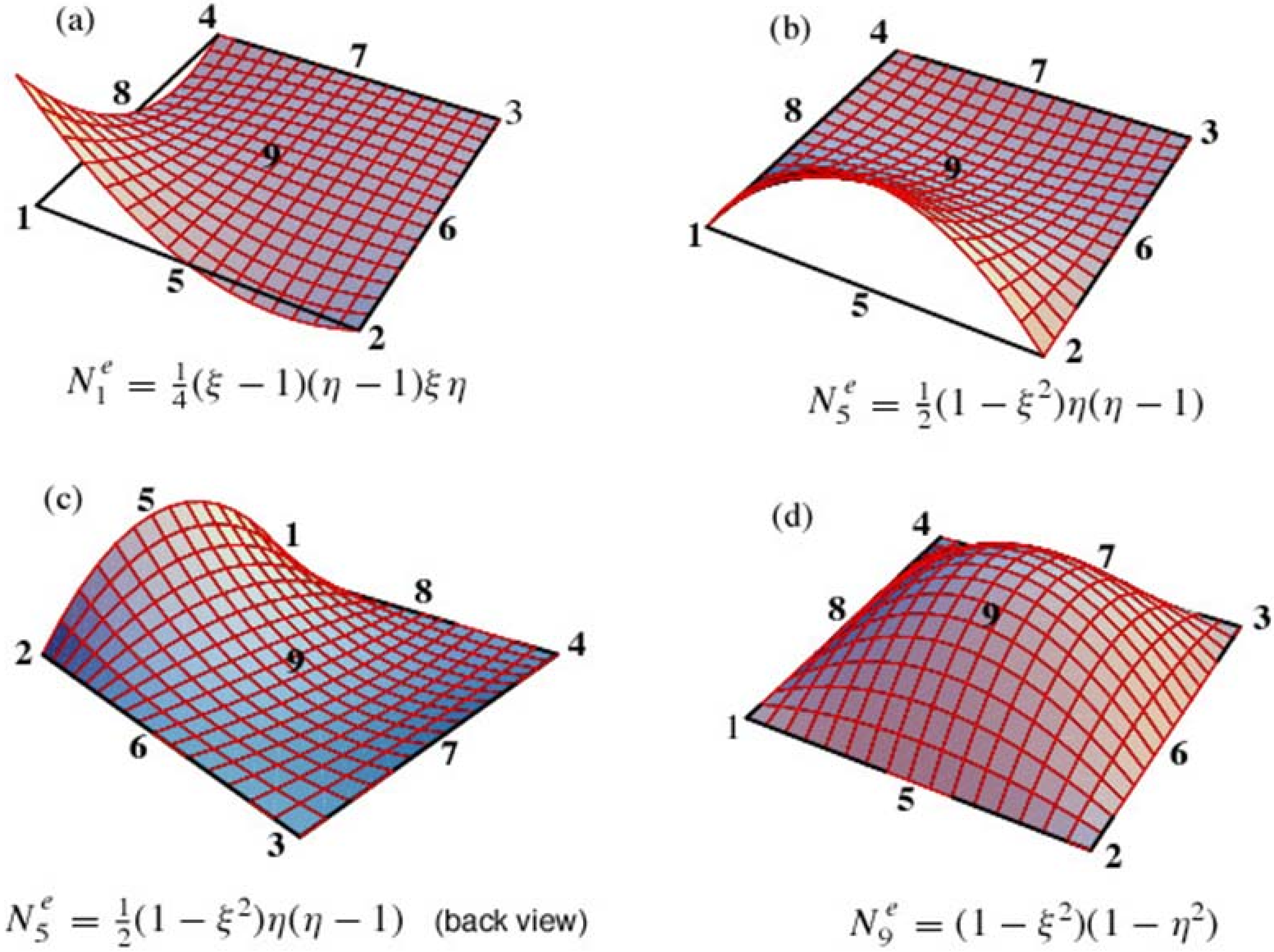
$$N^{(2)} = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N^{(3)} = \frac{1}{4}(1+\xi)(1+\eta)$$

$$N^{(4)} = \frac{1}{4}(1-\xi)(1+\eta)$$

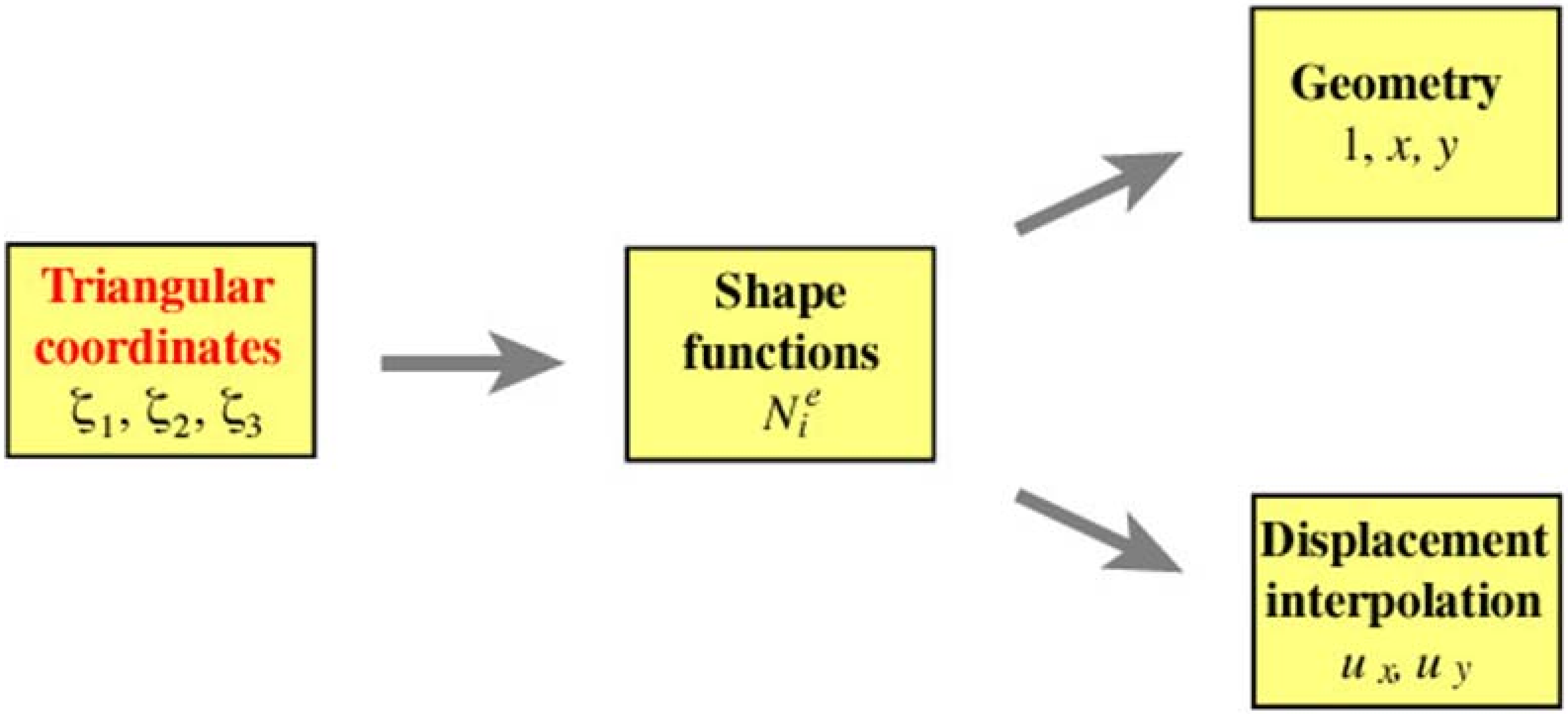


9 Node Biquadratic Quadrilateral Shape Function Plots



Triangular Elements

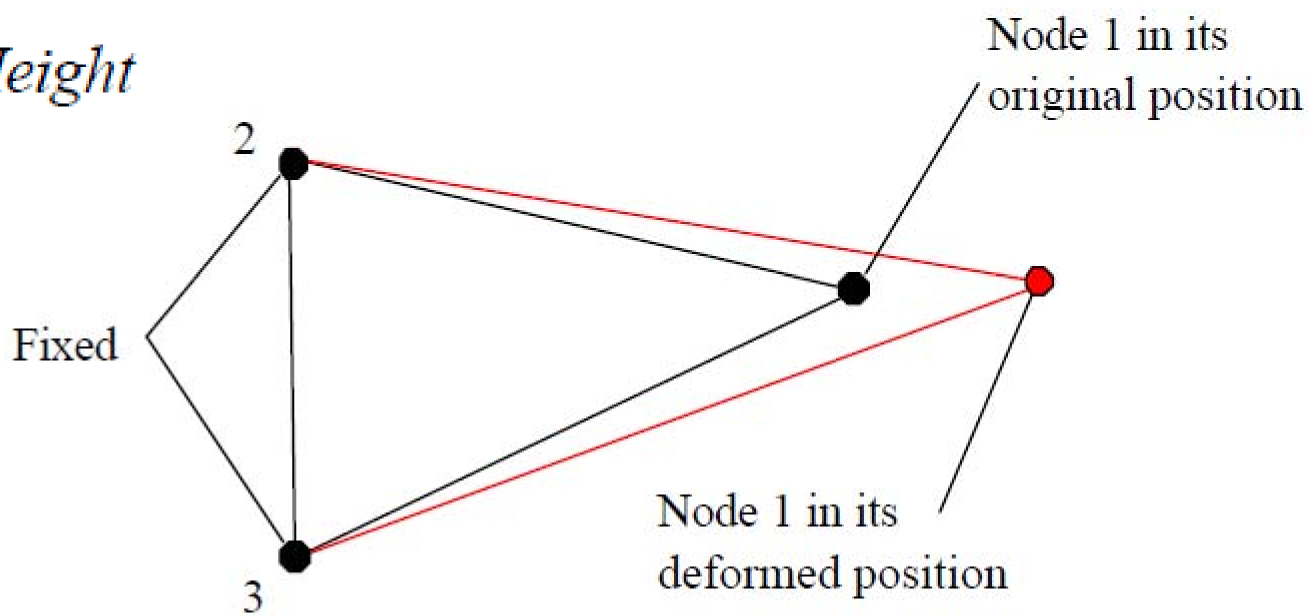
- Isoparametric Representation for Triangular



Triangular Elements

- From the diagram below, it is easy to see that points near nodes 2 and 3 will not move as far as points near node 1 when the triangle deforms. We will assume the deformation is linear and we will compute it with areas. The area of a triangle is;

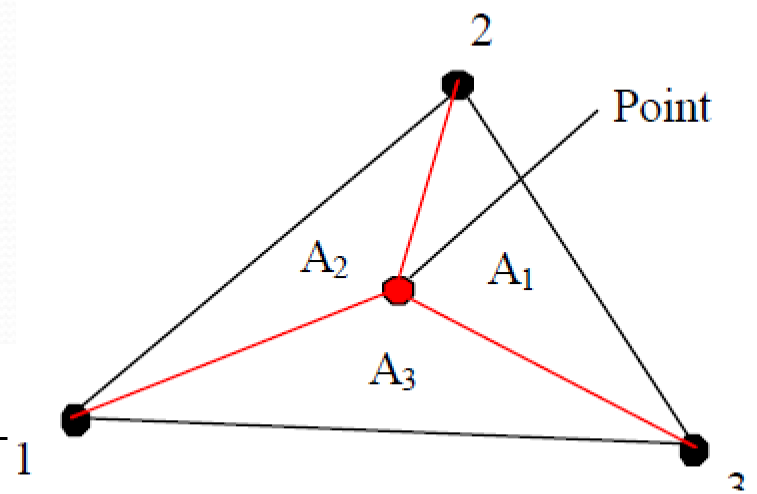
$$Area = \frac{1}{2} Base \times Height$$



Triangular Elements

- The interior point divides the triangle into 3 regions. All 3 nodal points may move and the motion of the interior point is some combination of their displacement. Let A1, A2, and A3 be the areas of each of triangular regions and A the total area of the element. We can see from the diagram that;

$$A = A_1 + A_2 + A_3$$



We can derive shape functions;

$$N_1 = \frac{A_1}{A},\ N_2 = \frac{A_2}{A},\ \text{and}\ N_3 = \frac{A_3}{A}$$

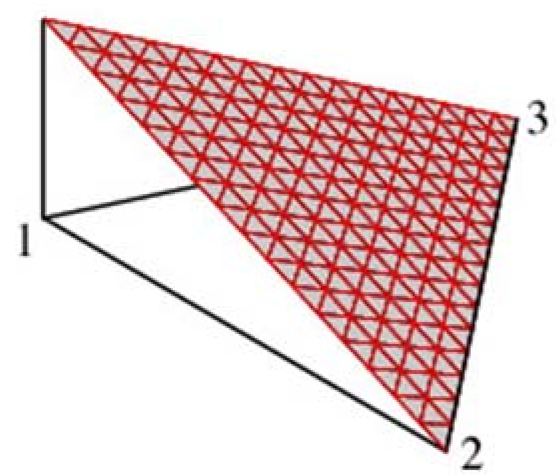
Triangular Elements

- The shape functions are not independent of one another because:

$$N_1 + N_2 + N_3 = 1$$

➢Knowing two of the shape functions makes it possible to compute the third. Because of this we can let

$$N_1 = \xi \,,\; N_2 = \eta \,, \text{and}\; N_3 = 1 - \xi - \eta$$



Triangular Elements

- **6 Node Triangle:Shape Function Plots**