Computational Engineering

Galerkin Method

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Galerkin Method

- Engineering problems: differential equations with boundary conditions.

  Generally denoted as: D(U)=0; B(U)=0
- Our task: to find the function U which satisfies the given differential equations and boundary conditions.
- Reality: difficult, even impossible to solve the problem analytically

Galerkin Method

- In practical cases we often apply approximation.
- One of the approximation methods:

  Galerkin Method, invented by Russian mathematician Boris Grigoryevich Galerkin.

Galerkin Method
Related knowledge

- Inner product of functions
- Basis of a vector space of functions

Galerkin Method
Inner product

- Inner product of two functions in a certain domain:

$$< f, g >= \int_a^b f(x)g(x)dx$$

shows the inner product of f(x) and g(x) on the interval [a, b].

*One important property: **orthogonality**

If $< f, g >= 0$, f and g are orthogonal to each other;

**If for arbitrary w(x), $< w, f >$ =0, f(x) $\equiv$ 0

Galerkin Method
Basis of a space

- V: a function space
- Basis of V: a set of linear independent functions $S = \{\phi_i(x)\}_{i=0}^{\infty}$

  Any function $f(x) \in V$ could be uniquely written as the linear combination of the basis:

$$f(x) = \sum_{j=0}^{\infty} c_j \phi_j(x)$$

Galerkin Method
Weighted residual methods

- A weighted residual method uses a finite number of functions $\{\phi_i(x)\}_{i=0}^{n}$.
- The differential equation of the problem is D(U)=0 on the boundary B(U), for example:

$D(U) = L(U(x)) + f(x) = 0$ on B[U]=[a,b].

where "L" is a differential operator and "f" is a given function. We have to solve the D.E. to obtain U.

Galerkin method
Weighted residual

- Step 1.

  Introduce a "trial solution" of U:

$$U \approx u(x) = \phi_0(x) + \sum_{j=1}^{n} c_j \phi_j(x)$$

to replace U(x)

$\phi_j(x)$: finite number of basis functions

c_j : unknown coefficients

* Residual is defined as:

$$R(x) = D[u(x)] = L[u(x)] + f(x)$$

Galerkin Method
Weighted residual

- Step 2.

  Choose "arbitrary" "weight functions" w(x), let: $< w, R(x) >=< w, D(u) > \int_a^b w(x)\{D[u(x)]\}dx = 0$

  With the concepts of "inner product" and "orthogonality", we have:

  The inner product of the weight function and the residual is zero, which means that the trial function partially satisfies the problem.

  So, our goal: to construct such u(x)

Galerkin Method
Weighted residual

- Step 3.

 Galerkin weighted residual method: choose weight function w <u>from the basis functions ϕ_j</u> , then

$$< w, R >= \int_a^b \phi_j [D(u)] dx = \int \phi_j (x) \{ D[\phi_0 (x) + \sum_{j=1}^{n} c_j \phi_j (x)] \} dx = 0$$

 These are a set of n-order linear equations. Solve it, obtain all of the coefficients c_j .

Galerkin Method
Weighted residual

- Step 4.

  The "trial solution" $u(x) = \phi_0(x) + \sum_{j=1}^{n} c_j \phi_j(x)$

  is the approximation solution we want.

Galerkin Method Example

- Solve the differential equation:

$$D(y(x)) = y''(x) + y(x) + 2x(1-x) = 0$$

with the boundary condition:

$$y(0) = 0, y(1) = 0$$

Galerkin Method Example

- Step 1.

  Choose trial function: $y(x) = \phi_0(x) + \sum_{i=1}^{n} c_i \phi_i(x)$

  We make n=3, and

  $\phi_0 = 0,$

  $\phi_1 = x(x-1),$

  $\phi_2 = x^2(x-1)^2$

  $\phi_3 = x^3(x-1)^3$

Galerkin Method Example

- Step 2.

  The "weight functions" are the same as the basis functions ϕ_i

  Step 3.

  Substitute the trial function y(x) into

$$< w, R > = \int_a^b \phi_j [D(u)] dx = \int \phi_j(x) \{ D[\phi_0(x) + \sum_{j=1}^{n} c_j \phi_j(x)] \} dx = 0$$

Galerkin Method Example

- Step 4.

  i=1,2,3; we have three equations with three unknown coefficients c_1, c_2, c_3

$$-\frac{1}{15}-\frac{3c_1}{10}+\frac{5c_2}{84}-\frac{4c_3}{315}=0$$

$$\frac{1}{70}+\frac{5c_1}{84}-\frac{11c_2}{630}+\frac{61c_3}{13860}=0$$

$$-\frac{1}{315}-\frac{4c_1}{315}+\frac{61c_2}{13860}-\frac{73c_3}{60060}=0$$

Galerkin Method Example

- Step 5.

Solve this linear equation set, get:

$$c_1 = -\frac{1370}{7397} \approx -0.18521$$

$$c_2 = \frac{50688}{273689} \approx 0.185203$$

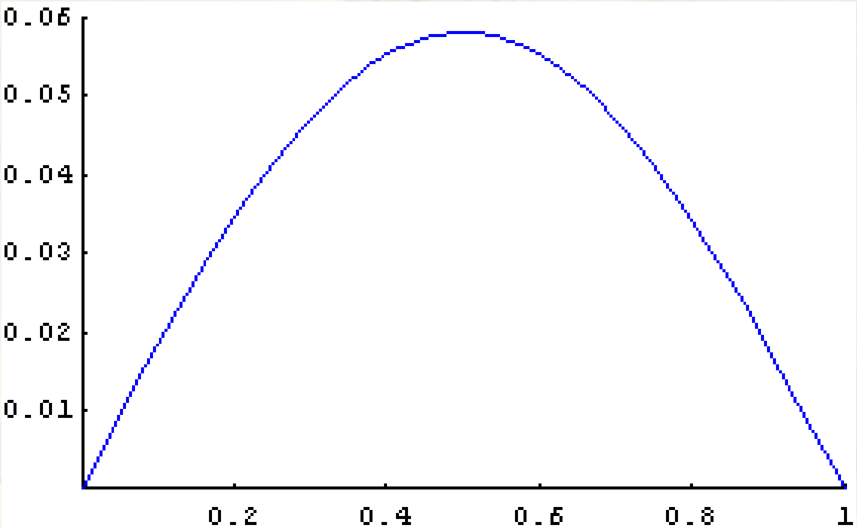
$$c_3 = -\frac{132}{21053} \approx -0.00626989$$

Obtain the approximation solution

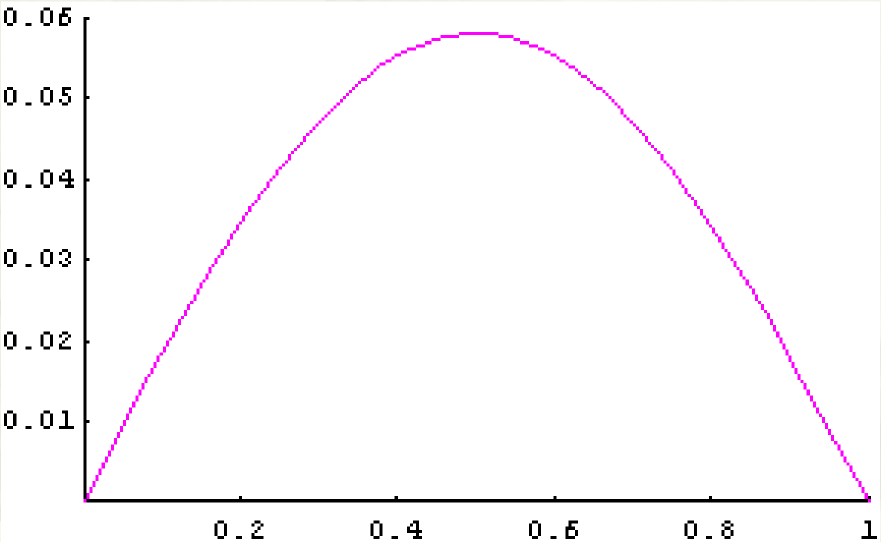
$$y(x) = \sum_{i=1}^{3} c_i \phi_i(x)$$

Galerkin Method Example

Galerkin solution



Analytic solution



References

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- 2. Galerkin method, Wikipedia: http://en.wikipedia.org/wiki/Galerkin_method#cite_note-BrennerScott-1